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Metric Regularity of the Feasible Set Mapping in Semi-Infinite Optimization

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Abstract. For parametric systems of (finitely many) equations and (infinitely many) inequalities the well-known concept of metric regularity is shown to be equivalent to the so-called extended Mangasarian–Fromovitz constraint qualification. By this, a corresponding result obtained by Robinson for finite optimization problems my be transferred to semi-infinite optimization. For the proof a local epigraph representation of the constraint set is mainly used.

Key Words. Semi-infinite optimization, Extended Mangasarian–Fromovitz constraint qualification, Metric regularity, Epigraph form of constraints.

AMS Classification. 90C30, 49D39, 90C31.

1. Introduction

Given a family $\Lambda(\cdot, t)$ $(t \in T)$ of multifunctions, a fixed $t^0 \in T$, and some (x^0, y^0) belonging to the graph of $\Lambda(\cdot, t^0)$, the notion of metric regularity indicates the existence of a neighborhood W of (x^0, y^0, t^0) such that, for each $(\bar{x}, \bar{y}, \bar{t}) \in W$, the set $\Lambda(\cdot, \bar{t})^{-1}(\bar{y}) := \{x \mid \bar{y} \in \Lambda(x, \bar{t})\}$ is nonempty and an *a priori* estimation of the distance between $\bar{x}$ and $\Lambda(\cdot, \bar{t})^{-1}(\bar{y})$ may be given in terms of the distance between $\bar{y}$ and $\Lambda(\bar{x}, \bar{t})$.

A standard application in optimization theory concerns studies of the regularity of the feasible set mapping for a parametrized system of finitely many equations and inequalities with C^1 data. Robinson [14], [15] has clarified the close connections—in fact, equivalences under suitable assumptions—between constraint qualifications (CQ) (Mangasarian–Fromovitz CQ, Slater CQ, Robinson CQ, etc.) and metric regularity in the classical settings of nonlinear programming problems; for a recent paper which completes the studies in this context, we refer to [5]. Among the large number of publications handling metric regularity of multifunctions on different levels of generality we also mention those of Ioffe [9], Zowe and Kurcyusz [18], Penot [13], Aubin [2], Auslender [3], Rockafellar [17], Borwein [4], and Jourani and Thibault [11]. Metric regularity of the constraint set mapping plays an important role in the stability and sensitivity analysis of parametric nonlinear programs, see, e.g., [1]–[3], [5], [6], [12], [16], and [17].

This paper is concerned with conditions ensuring metric regularity of the feasible set mapping of a semi-infinite program. We extend Robinson's result [14] that the Mangasarian–Fromovitz CQ is, for standard finite nonlinear programs, equivalent to the regularity of the constraints. Thereby we relate to a CQ introduced by Jongen *et al.* [10] who pointed out its relevance concerning structural stability in semi-infinite optimization. As a main tool we use a technique proposed in [7] to represent locally the constraints in epigraph form leading to a very simple realization of metric regularity.

2. Metric Regularity Results

Consider the following parametric system of equations and inequalities, which typically appears as constraint system in semi-infinite optimization problems:

$$h_i(x, t) = u_i \quad (i = 1, \ldots, p) \quad \text{and} \quad g(j, x, t) \geq v(j) \quad (\forall j \in K), \tag{1}$$

where $t \in T$, $u = (u_1, \ldots, u_p) \in \mathbb{R}^p$, and $v \in C(K, \mathbb{R})$ are viewed as parameters, K is a compact subset of $\mathbb{R}^s$, T is a metric space, and h_i: $\mathbb{R}^n \times T \to \mathbb{R}$ and g: $\mathbb{R}^s \times \mathbb{R}^n \times T \to \mathbb{R}$ are given functions. $C(K, \mathbb{R})$ denotes the space of continuous functions from K to R with norm $\|v\|_K := \max_{j \in K}|v(j)|$. For each (x, t) let

$$h(x, t) := (h_1(x, t), \ldots, h_p(x, t)) \quad \text{and} \quad g(\cdot, x, t)(j) := g(j, x, t) \ (\forall j \in K).$$

Throughout the paper we impose the following continuity properties.

Assumption 1. The constraint functions satisfy:

1. g and h_i are continuous.
2. g is continuously differentiable with respect to (j, x), and h is continuously differentiable with respect to x.
3. The partial derivatives $D_{(j,x)}g(\cdot, \cdot, \cdot)$ and $D_x h_i(\cdot, \cdot)$ are continuous.

Obviously, by condition 1 of Assumption 1, the inequalities in (1) may be interpreted as a cone constraint in $C(K, \mathbb{R})$, namely, $g(\cdot, x, t) - v(\cdot) \in C_0(K)$, where $C_0(K) := \{v \in C(K, \mathbb{R}) \mid v(j) \geq 0 \ (\forall j \in K)\}$.

Specializing the notions of metric regularity in Definition 3.1 of [11] and Definition 2.1, parametric version, of [5] to our situation, we arrive at the following definition. Let $\|\cdot\|_\infty$ denote the l_∞-norm. For $u \in \mathbb{R}^p$, $v \in C(K, \mathbb{R})$, and $\alpha \in \mathbb{R}$ put $\|(u, v)\|_{\infty,K} := \max\{\|u\|_\infty, \|v\|_K\}$, $\alpha^+ := \max\{\alpha, 0\}$ and let v^+ be defined by $v^+(j) := (v(j))^+$, $\forall j \in K$. By *dist* we denote the point-to-set distance induced by any norm $\|\cdot\|$ in $\mathbb{R}^n$. Let 0_p and θ be the zero elements in $\mathbb{R}^p$ and $C(K, \mathbb{R})$, respectively. For $\omega = (t, u, v) \in T \times \mathbb{R}^p \times C(K, \mathbb{R})$ we denote by $M(\omega) \subseteq \mathbb{R}^n$ the solution set of (1).

Let $\omega^0 = (t^0, 0_p, \theta)$ and $x^0 \in M(\omega^0)$. We call system (1) *metrically regular* at (x^0, ω^0) if a neighborhood U of (x^0, ω^0) and a real number $\beta > 0$ exist such that, with $\omega = (t, u, v)$,

$$dist(x, M(\omega)) \leq \beta \cdot \|(u - h(x, t), (v - g(\cdot, x, t))^+)\|_{\infty,K}, \qquad \forall (x, \omega) \in U. \tag{2}$$

Note that (2) includes that $M(\omega)$ is nonempty for all ω in some neighborhood of ω^0. When fixing $t = t^0$ in this definition, we call system (1) *metrically regular* at (x^0, ω^0) *with respect to right-hand side perturbations.* Defining the multifunction $\Lambda(x, t) := \{(h(x, t), g(\cdot, x, t))\} - \{0_p\} \times C_0(K)$, metric regularity in the sense of (2) fits into the notion introduced in the very beginning of this paper.

For a feasible point $x \in M(t^0, 0_p, \Theta)$, let $E(x) := \{j \in K \,|\, g(j, x, t^0) = 0\}$. Clearly, compactness of K implies compactness of $E(x)$. Note that, due to feasibility of x, the set $E(x)$ consists of all global minima of the function $g(\cdot, x, t^0)$ on K if $E(x) \neq \emptyset$.

Following [10] (see also p. 47 of [8]) we say that the *Extended Mangasarian–Fromovitz Constraint Qualification* (EMFCQ) holds at $x^0 \in M(t^0, 0_p, \Theta)$ if

$$\{D_x h_i(x^0, t^0)\}_{i=1,\dots,p} \text{ is a linearly independent set} \tag{3}$$

and

$$\exists \xi \in \mathbb{R}^n, \quad D_x h_i(x^0, t^0)\xi = 0 \;\; (i = 1, \dots, p), \quad D_x g(j, x^0, t^0)\xi > 0 \;\; (\forall j \in E(x^0)). \tag{4}$$

If K is a finite set, EMFCQ is the standard Mangasarian–Fromovitz CQ. Obviously, EMFCQ implies $p \leq n$ and, if $E(x^0) \neq \emptyset$, then $\xi \neq 0_n$ and $p \leq n - 1$.

Now we present the results of this paper. The idea of our approach relies essentially on the following observation for a nonparametric inequality system: If it is assumed that t^0 is fixed, no equations appear, and, for each $j \in K$, $g(j, x, t^0)$ has the form $x_n - \tilde{g}(j, x_1, \dots, x_{n-1})$, then the distance between some point and the feasible set can be trivially estimated along the x_n-axis. In the case of the general system (1), estimate (2) reduces to a similarly simple form if a parametric epigraph representation exists. As the subsequent lemma will show, EMFCQ provides this after some local coordinate transformation. The proof of this lemma is almost identical to that of the nonparametric version [7] and is therefore omitted.

For the purpose of abbreviation we introduce the projection $[y]_k := (y_1, \dots, y_k)$ of a point $y \in \mathbb{R}^n$ onto its first k coordinates.

Lemma 1. *Assume* EMFCQ *to hold at a point* $x^0 \in M(t^0, 0_p, \Theta)$ *with* $E(x^0) \neq \emptyset$. *Then an open neighborhood* $U_x \times U_u \times U_v \times U_y$ *of* $(x^0, 0_p, \Theta, 0_n)$, *an open set*

$\mathcal{O} \supseteq E(x^0)$, and a local C^1-coordinate transformation $\psi: U_x \to U_y$ with $\psi(x^0) = 0_n$ exist such that, for all $(x, u, v, y) \in U_x \times U_u \times U_v \times U_y$, $y = \psi(x)$, the following systems are equivalent:

$$\left\{\begin{array}{c} h(x, t^0) = u \\ g(j, x, t^0) \geq v(j), \quad \forall j \in K \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} [y]_p = u \\ y_n \geq \varphi(j, [y]_{n-1}, v), \quad \forall j \in K^* \end{array}\right\}.$$

Here, K^* is a compact set with $E(x^0) \subseteq K^* \subseteq K \cap \mathcal{O}$ and $\varphi: \mathcal{O} \times [U_y]_{n-1} \times U_v \to \mathbb{R}$ is continuously differentiable on its domain. Further, a real number $\alpha > 0$ exists such that $|\varphi(j, [y]_{n-1}, v) - y_n| \leq \alpha \cdot |v(j) - g(j, x, t^0)|$, $\forall (j, x, v, y) \in K^* \times U_x \times U_v \times U_y$, $y = \psi(x)$.

In what follows let $t^0 \in T$ and $\omega^0 = (t^0, 0_p, \theta)$. For brevity we sometimes write $h(x)$, $g(j, x)$, $M(u, v)$ instead of $h(x, t^0)$, $g(j, x, t^0)$, $M(t^0, u, v)$.

Theorem 1. *If* EMFCQ *holds at $x^0 \in M(\omega^0)$, then system* (1) *is metrically regular at (x^0, ω^0).*

Proof. Relating a result given by Cominetti in Theorem 2.1 of [5] to our situation means that metric regularity of (1) with respect to right-hand side perturbations with some modulus $\beta > 0$ implies metric regularity of (1), if the functions $x \mapsto (h(x, t) - h(x, t^0))$ and $x \mapsto (g(\cdot, x, t) - g(\cdot, x, t^0))$ are Lipschitzian on some neighborhood of x^0 with uniform Lipschitz constant $L < \beta^{-1}$ for all t in some neighborhood of t^0. This, however, follows from condition 3 of Assumption 1 and the compactness of K by standard arguments. Hence it suffices to show that (1) is metrically regular at (x^0, ω^0) with respect to right-hand side perturbations.

In the case $E(x^0) = \emptyset$, condition 1 of Assumption 1 yields that in some neighborhood of (x^0, ω^0), system (1) is exclusively described by the equations. However, then the classical MFCQ in finite optimization is dealt with and the stated proposition follows from [14]. For $E(x^0) \neq \emptyset$ Lemma 1 may be applied. We consider the system

$$[y]_p = u \qquad \text{and} \qquad y_n \geq \varphi(j, [y]_{n-1}, v), \quad \forall j \in K^*, \tag{5}$$

in the setting of Lemma 1. In order to verify metric regularity, let $(\bar{x}, u, v, \bar{y})$ in $U_x \times U_u \times U_v \times U_y$ with $\bar{y} = \psi(\bar{x})$ be arbitrarily chosen. Define $M^*(u, v) := \{y \in U_y \mid y \text{ satisfies (5)}\}$. Clearly, $M^*(u, v) = \psi(U_x \cap M(u, v))$, by Lemma 1. Define a reference point

$$y^* \in M^*(u, v) \text{ by } [y^*]_{n-1} := (u, \bar{y}_{p+1}, \ldots, \bar{y}_{n-1}) \quad \text{and} \quad y_n^* := \varphi^*([y^*]_{n-1}, v),$$

where $\varphi^*([y]_{n-1}, v) := \max_{j \in K^*} \varphi(j, [y]_{n-1}, v)$. Feasibility of y^* follows from Lemma 1. Then

$$\begin{aligned} dist(\bar{y}, M^*(u, v)) &\leq c \cdot \|\bar{y} - y^*\|_\infty \\ &= c \cdot \max\{\|[\bar{y}]_p - u\|_\infty, |\bar{y}_n - \varphi^*([y^*]_{n-1}, v)|\}, \end{aligned} \tag{6}$$

where c is some positive factor of norm equivalence.

Completing the proof, we first remark that $x^* := \psi^{-1}(y^*) \in M(u, v)$. Secondly, recall that ψ^{-1} is continuously differentiable on U_y and may be hence assumed, without loss of generality, to be Lipschitzian there with modulus $\varepsilon > 0$. From (6) we conclude (with $\bar{y} = \psi^{-1}(\bar{x})$)

$$\begin{aligned} dist(\bar{x}, M(u, v)) &\leq dist(\bar{x}, U_x \cap M(u, v)) \\ &= \inf_{y \in M^*(u,v)} \|\psi^{-1}(\bar{y}) - \psi^{-1}(y)\| \\ &\leq \varepsilon \cdot dist(\bar{y}, M^*(u, v)) \\ &\leq c \cdot \varepsilon \cdot \max\{\|[\bar{y}]_p - u\|_\infty, |\varphi^*([y^*]_{n-1}, v) - \bar{y}_n|\}. \end{aligned} \tag{7}$$

Without loss of generality, the maximum function φ^* may be assumed to be Lipschitzian on $[U_y]_{n-1} \times U_v$ with some modulus $\delta > 0$. Taking account of the second assertion in Lemma 1 we thus obtain

$$\begin{aligned} &|\bar{y}_n - \varphi^*([y^*]_{n-1}, v)| \\ &\quad \leq |\varphi^*([y^*]_{n-1}, v) - \varphi^*([\bar{y}]_{n-1}, v)| + |\varphi^*([\bar{y}]_{n-1}, v) - \bar{y}_n| \\ &\quad \leq \delta \cdot \|u - [\bar{y}]_p\|_\infty + \alpha \cdot \|(v - g(\cdot, \bar{x}))^+\|_K. \end{aligned}$$

Combining this with (7) yields the required condition of metric regularity with modulus $\beta := c \cdot \varepsilon \cdot \max\{1, \alpha + \delta\} > 0$. □

Theorem 2. *If system* (1) *is metrically regular at* $x^0 \in M(\omega^0)$ *with respect to right-hand side perturbations, then EMFCQ holds at* x^0.

Proof. For $\varepsilon \in \mathbb{R}$ let $\hat{\varepsilon}$ be the function defined by $\hat{\varepsilon}(j) = \varepsilon$ $(\forall j \in K)$. By hypothesis, neighborhoods U of 0_p, V of x^0, and positive real numbers β, γ exist such that, for all $(x, u) \in V \times U$ and all $\varepsilon \in [0, \gamma]$,

$$dist(x, M(u, -\hat{\varepsilon})) \leq \beta \cdot \|(u - h(x), (-\hat{\varepsilon} - g(\cdot, x))^+)\|_{\infty, K}.$$

The compactness of K and the continuity of g entail that, for some neighborhood $V' \subseteq V$ of x^0, $g(j, x) \geq -\gamma$ $(\forall (j, x) \in V' \times U)$. Hence, we have for the solution set mapping $u \to \tilde{M}(u)$ of the system $h(x) = u$ that if $(x, u) \in V' \times U$, then

$$\begin{aligned} dist(x, \tilde{M}(u)) &\leq dist(x, M(u, -\hat{\gamma})) \\ &\leq \beta \cdot \max\left\{\|u - h(x)\|_\infty, \max_{j \in K}(-\gamma - g(j, x))^+\right\} \\ &= \beta \cdot \|u - h(x)\|_\infty, \end{aligned}$$

which implies by classical regularity theory (see, e.g., Corollary 3 of [14]) that (3) is satisfied.

Now we prove (4). If $E(x^0) = \emptyset$, then locally only equality constraints are dealt with, but then the assertion follows from [14] again. Therefore let $E(x^0) \neq \emptyset$.

Then $\min_{j\in K} g(j, x^0, t^0) = 0$. Consider a sequence $\varepsilon_k \downarrow 0$ with $\varepsilon_k > 0$. Since $(u, v) = (0_p, \hat{\varepsilon}_k)$ are feasible right-hand side perturbations of (1), metric regularity implies that to each ε_k there belongs a point $x^k \in M(t^0, 0_p, \hat{\varepsilon}_k)$ fulfilling

$$\|x^k - x^0\| = dist(x^0, M(t^0, 0_p, \hat{\varepsilon}_k)) \le \beta \cdot \|(\hat{\varepsilon}_k - g(\cdot, x^0, t^0))^+\|_K = \beta \cdot \varepsilon_k. \tag{8}$$

For all $j \in E(x^0)$ we have $g(j, x^0, t^0) = 0$ and, due to feasibility of x^k, we have $g(j, x^k, t^0) \ge \varepsilon_k$. Taking account of (8), this leads to $(\forall j \in E(x^0))$

$$\begin{aligned}\frac{1}{\beta} \le \frac{\varepsilon_k}{\|x^k - x^0\|} &\le \frac{g(j, x^k, t^0) - g(j, x^0, t^0)}{\|x^k - x^0\|} \\ &= \frac{D_x g(j. x^0, t^0)(x^k - x^0) + o(x^k - x^0)}{\|x^k - x^0\|}.\end{aligned} \tag{9}$$

By (8) $\lim_{k\to\infty} \|x^k - x^0\| = 0$ holds. Without loss of generality we may assume $\lim_{k\to\infty} \|x^k - x^0\|^{-1}(x^k - x^0) = \xi \in \mathbb{R}^n$. Transition to the limits on both sides of (9) provides $D_x g(j, x^0, t^0) \cdot \xi \ge (1/\beta) > 0$ $(\forall j \in E(x^0))$. On the other hand, since $x^0 \in M(t^0, 0_p, \Theta)$ and $x^k \in M(t^0, 0_p, \hat{\varepsilon}_k)$, it follows that $h(x^0, t^0) = h(x^k, t^0) = 0_p$, hence $Dh_i(x^0, t^0)(x^k - x^0) = o(x^k - x^0)$. Dividing by $\|x^k - x^0\|$ and passing to the limits yield $Dh_i(x^0, t^0) \cdot \xi = 0$ $(i = 1, \ldots, p)$. □

3. Concluding Remarks

By Theorems 1 and 2 we may conclude that, for parametric systems of the type (1), the concept of metric regularity is equivalent to EMDCQ. This fact could be alternatively proved by applying results in [5], [13], and [18] to the cone constraint form of (1), but we have preferred a more direct approach. The results given above enlarge the set of equivalent characterizations of EMFCQ. We emphasize that in [10] EMFCQ was found to coincide with stability of constraint sets in semi-infinite optimization. By this it is recognized that metric regularity is closely related to stability in the sense of Jongen. The difference between both concepts is based on somewhat different types of data perturbations. Finally, in [7] equivalence between EMFCQ and local epigraph representability of the constraint set was shown.

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