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REGULARITY AND STABILITY IN NONLINEAR SEMI-INFINITE OPTIMIZATION

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ABSTRACT

The paper is concerned with semi–infinite C^1 programs parametrized in the objective function and in the constraint functions, where perturbations may also occur in the index set of the semi–infinite constraints. Our purpose is to give a self–contained presentation of the interrelations between metric regularity, extended Mangasarian–Fromovitz constraint qualification, local boundedness of multipliers and upper semicontinuity of stationary solutions. Moreover, we outline stability properties of perturbed local minimizers in the absence of second–order differentiability of the data.

1 INTRODUCTION

In this paper, we are concerned with parametric nonlinear programs involving finitely many equality and infinitely many inequality constraints in finite dimensions. The parameters may occur both in the problem functions and in the index set associated with the inequalities. Our studies will essentially be restricted to the use of first–order information.

Consider a family of semi-infinite optimization problems

$$\mathrm{P}(t): \qquad f(t,x) \to \min_x \qquad \text{s.t.} \qquad x \in M(t)\,,$$

where t is a parameter varying over T, and M is a multifunction which assigns to each $t \in T$ the solution set $M(t)$ of the system

$$\begin{aligned} h_i(t,x) &= 0 \qquad (i=1,...,p) \\ g(t,x,y) &\geq 0 \qquad (\forall y \in K(t)). \end{aligned} \tag{1.1}$$

Throughout the paper we shall make the following

General Assumptions (GA). T is a metric space, $\mathcal{K}$ is a nonempty and compact subset of $\mathbb{R}^s$, and the functions $f : T \times \mathbb{R}^n \to \mathbb{R}$, $h = (h_1, ..., h_p) : T \times \mathbb{R}^n \to \mathbb{R}^p$ and $g : T \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ as well as the multifunction $K : T \to \mathcal{K}$ have the following properties:

$$f, h \text{ and } g \text{ are continuous,} \tag{1.2}$$

$$f, h \text{ and } g \text{ are differentiable with respect to } x, \tag{1.3}$$

$$\nabla_x f(.,.), \nabla_x h(.,.) \text{and } \nabla_x g(.,.,.) \text{ are continuous,} \tag{1.4}$$

$$\emptyset \neq K(t) \subset \mathcal{K} \quad \text{and} \quad K(t) \text{ is compact} \qquad (\forall t \in T). \tag{1.5}$$

$$t \mapsto K(t) \text{ is closed and lower semicontinuous on } T, \tag{1.6}$$

where $\nabla_x f(\cdot,\cdot)$, $\nabla_x h(\cdot,\cdot)$ and $\nabla_x g(\cdot,\cdot,\cdot)$ denote the gradient mappings of f, h, g with respect to x. For further notation and definitions we refer to the end of Section 1.

The purposes of our paper are, on the one hand, to study the close interrelations between metric regularity, extended Mangasarian–Fromovitz constraint qualification, local boundedness of multipliers and upper semicontinuity of stationary solutions, and, on the other hand, to discuss certain consequences for the stability behavior of perturbed local minimizers in the absence of second–order differentiability assumptions on the data. Since results of these types are scattered over the literature, we strive for a presentation which is as much as possible self–contained. For other views of perturbation and stability analysis in semi–infinite programming, we refer to the papers contributed by Jongen and Rückmann [23] and Shapiro [55] to the present volume.

Section 2 which essentially goes back to the first author's paper [34] is devoted to generalizations of Gauvin's theorem [14] and of the parametric version of Gauvin's theorem (Robinson [46]). These results were originally given for standard nonlinear programs and say (in the parametric variant) that the Mangasarian–Fromovitz constraint qualification (MFCQ) holds at some stationary solution of the unperturbed problem if and only if the associated Lagrange multiplier set mapping of the parametric problem is locally bounded. Now, in the semi–infinite programming setting, MFCQ has to be replaced by a standard extension of it. We also outline the relations to constraint qualifications and Gauvin type theorems for other settings of semi–infinite programs.

Section 3 extends the authors' [16] results given for $\mathrm{P}(t)$ in the case of a fixed index set $K(t) \equiv K$ to a parametric index set $K(t)$, t varying. We shall show that metric regularity of the constraints near some feasible point is equivalent to satisfying the extended MFCQ at this point. As an essential prerequisite, we use Cominetti's lemma [11] on the relation between the metric regularities under arbitrary perturbations and right–hand side perturbations of the constraints. We shall use the fact that the extended MFCQ allows an equivalent epigraphical representation of the constraints, see [15, 16]. An alternative proof using the cone constraint approach could be given by combining results from [11, 43, 54]. Finally, in the case of absence of equations in (1.1), we characterize metric regularity by an explicit growth condition of the constraint function, thereby we follow an idea in [17].

Section 4 treats some consequences of Section 3 for the stability behavior of local minimizers. The approach is based – for the qualitative stability part – on [6, 29, 47] and – for the quantitative stability part - on an idea of Alt [1] and modifications of this idea given in [5, 29, 30, 32]. Some Concluding Remarks (Section 5) will complete the paper.

We finish Section 1 by introducing some definitions and notation which will be used throughout the paper. Notation and definitions needed only locally will be found in the corresponding (sub)section. In $\mathbb{R}^n$ we denote by 0_n the zero vector, by $\langle x, y\rangle$ or $x^{\mathrm{T}}y$ the Euclidean inner product, by $\|\cdot\|$ the Euclidean norm, by $\|\cdot\|_\infty$ the maximum norm. By "$\operatorname{dist}(x, Z)$" we mean the usual distance of a point x to a set Z induced by the norm $\|\cdot\|$ in $\mathbb{R}^n$. We write $\mathbb{R}^n_+$ for the non-negative orthant of $\mathbb{R}^n$, and use the abbreviation $v_{[r]}$ to denote the projection of an n–vector v onto its first r coordinates $(v_1, ..., v_r)$. By $\operatorname{conv} X$ and $\operatorname{cone} X$ we mean the convex and the conic hull of $X \subset \mathbb{R}^n$, respectively, while $\dim X$ refers to the dimension of a convex set X. We use the symbol $C(K, \mathbb{R})$ for the linear space of continuous functions $y \mapsto b(y)$ from the compact set $K \subset \mathbb{R}^s$ to $\mathbb{R}$, equipped with the norm $\|b\|_K := \max_{y \in K} |b(y)|$. The zero element of $C(K, \mathbb{R})$ will be denoted by Θ. For each $(t, x) \in T \times \mathbb{R}^n$ we define

$$E(t,x) := \{y \in K(t) \,|\, g(t,x,y) = 0\}, \quad K(t,x) := \{\nabla_x g(t,x,y) \,|\, y \in E(t,x)\}.$$

Both sets are compact due to the assumptions GA. Note that the set of active indices $E(t,x)$ consists of all global minimizers of the function $g(t,x,\cdot)$ on $K(t)$, provided that $E(t,x)$ is nonempty and $g(t,x,y) \geq 0$ for all $y \in K(t)$.

When $t \in T$ is given, a point $x \in \mathbb{R}^n$ is a (Karush-Kuhn-Tucker type) *stationary solution* of P(t), if there are a nonnegative integer $r \leq n+1$, a (possibly empty) subset $\{y^1, ..., y^r\}$ of $E(t,x)$ as well as multipliers $u_i \in \mathbb{R}$ $(i = 1, ...p)$ and $v_j \geq 0$ $(j = 1, ..., r)$ such that

$$x \in M(t), \quad \nabla_x f(t,x) + \sum_{i=1}^{p} u_i \nabla_x h_i(t,x) - \sum_{j=1}^{r} v_j \nabla_x g(t,x,y^j) \;=\; 0_n. \quad (1.7)$$

Given some $x \in M(t^0)$ for fixed $t = t^0$, we shall say that x satisfies the *Extended Mangasarian Fromovitz Constraint Qualification* (EMFCQ) with respect to (1.1) at $t = t^0$, if

$$\begin{aligned} &\nabla_x h(t^0,x) \text{ has rank } p, \text{ and there is some } \xi \in \mathbb{R}^n \text{ with} \\ &\xi^{\mathrm{T}} \nabla_x h_i(t^0,x) = 0 \ (i = 1, ..., p),\ \xi^{\mathrm{T}} \nabla_x g(t^0,x,y) > 0 \ (\forall y \in E(t^0,x)). \end{aligned} \quad (1.8)$$

The latter definition is due to Jongen, Twilt and Weber [26], it is based on a condition introduced in [18, 20]. Indeed, (1.8) is a constraint qualification assuring that a local minimizer of (P) is a stationary solution of (P), cf., e.g., [20, Satz 3.1.14]. Obviously, EMFCQ includes $p \leq n$ and, if $E(t^0,x) \neq \emptyset$ then $\xi \neq 0_n$ and $p \leq n-1$ hold under EMFCQ at $x \in M(t^0)$. If $K(t^0)$ is finite, Condition (1.8) is the standard Mangasarian-Fromovitz constraint qualification. We note that the case of having no equality constraints $(p = 0)$ is not excluded from the following results. For simplicity we admit $p = 0$ when speaking of p-tuples, (p,n)-matrices, p-sums etc., the corresponding term is then assumed being vacuous.

Often we consider unperturbed semi–infinite programs, i.e., we may suppose $t^0 \in T$ being fixed. For simplicity, we write in this situation

$$f(x) := f(t^0,x),\ h(x) := h(t^0,x),\ g(x,y) := g(t^0,x,y), \quad (1.9)$$

as well as $E(x) := E(t^0,x)$, $K := K(t^0)$, and $\Gamma(x) := \Gamma(t^0,x)$.

Finally, we introduce the continuity notions for multifunctions, as they will be used below, for more details we refer to [6, Section 2.2]. Given a multifunction Φ from a metric space $\mathcal{S}$ to $\mathbb{R}^d$, Φ is called *closed at* $s^0 \in \mathcal{S}$, if $s^k \to s^0$, $w^k \to w^0$ and $w^k \in \Phi(s^k)$ $(\forall k)$ imply that $w^0 \in \Phi(s^0)$. Φ is said to be *locally bounded at* $s^0 \in \mathcal{S}$ if for some neighborhood $\mathcal{O}$ of s^0 and some nonempty bounded set $Q \subset \mathbb{R}^d$ and for all $s \in \mathcal{O}$, $\Phi(s)$ is a subset of Q. Recall that for a closed multifunction, upper semicontinuity in Berge's sense is equivalent to local boundedness. So, in our context, it will be suitable to say that Φ is *upper semicontinuous at* s^0 if Φ is closed and locally bounded at s^0. If the metric space $\mathcal{S}$ is compact, then closedness and upper semicontinuity coincide. Moreover, Φ is said to be *lower semicontinuous at* (s^0, w^0) if, with $w^0 \in \Phi(s^0)$, for each sequence $s^k \to s^0$ there is some sequence $w^k \to w^0$ satisfying $w^k \in \Phi(s^k)$ for k sufficiently large. Φ is said to be *lower semicontinuous at* s^0 if Φ is lower semicontinuous at any point of $\{s^0\} \times \Phi(s^0)$. Any of these properties is said to hold *on* $\mathcal{S}$ if it holds at all s in $\mathcal{S}$.

2 UPPER SEMICONTINUITY OF STATIONARY POINTS

In Section 2 we give extensions of Gauvin's Theorem [14] of Robinson's [46] parametric variant of this result. In particular, it turns out that EMFCQ is equivalent to the local boundedness of the multiplier set mapping. This implies the upper semicontinuity of the stationary solution set mapping.

2.1 Preliminaries

We begin by discussing some basic concepts which are related to unperturbed semi–infinite programs. Hence, we may suppose $t^0 \in T$ being fixed and use the notation (1.9). The semi–infinite problem $\mathrm{P}(t^0)$ then reads

$$(\mathrm{P}): \quad f(x) \;\to\; \min \quad \text{s.t.} \quad x \in \mathcal{M}$$

where $\mathcal{M}$ is the solution set of the system

$$h_i(x) = 0 \, (i = 1, ..., p), \; g(x, y) \geq 0 \, (\forall y \in K). \tag{1.1}$$

The following Gordan-type lemma (for a proof see, e.g., [34]) gives a dual form of the extended Mangasarian–Fromovitz condition. We note that the statement of the lemma is standard if no equations appear, cf., e.g., Cheney [9].

Lemma 2.1 *Let Q be a nonempty compact subset of $\mathbb{R}^n$, and let B be a (p, n)-matrix. The following are equivalent:*

a. *B has rank p, and there is some* $\xi \in \mathbb{R}^n$ *such that*
$B\xi = 0_p \ , \quad q^{\mathrm{T}}\xi > 0\,(\forall q \in Q)$.

b. *For each set of* $\nu = n - p + 1$ *points* $\{q^1, ..., q^\nu\}$ *in* Q, *the system*
$B^{\mathrm{T}}u - Dv = 0_n \ , \quad 0_{p+\nu} \neq (u, v) \in \mathbb{R}^p \times \mathbb{R}^\nu_+$
has no solution, where D *is the* (n, ν)*-matrix* $D := [q^1 \dots q^\nu]$.

c. *For each* $r \geq 0$ *and each set of* r *points* $\{q^1, ..., q^r\}$ *in* Q, *the system*
$B^{\mathrm{T}}u - \tilde{D}v = 0_n \ , \quad 0_{p+r} \neq (u, v) \in \mathbb{R}^p \times \mathbb{R}^r_+$
has no solution, where $\tilde{D}$ *is the* (n, r)*-matrix* $\tilde{D} := [q^1 \dots q^r]$. ◇

Now we recall two other approaches of modelling the semi–infinite program (P): the cone constraint formulation which leads to an optimization problem in the Banach space of continuous functions over a compactum, and the maximum constraint formulation which leads to a Lipschitz program in finite dimensions.

For each $x \in \mathbb{R}^n$, let $g(x, \cdot)$ be the function defined by

$$g(x, \cdot)(y) := g(x, y) \quad (\forall y \in K),$$

hence, by the general assumptions, we have $g(x, \cdot) \in C(K, \mathbb{R})$ for all x. Let

$$C \ := \ \{0_p\} \times C_+(K, \mathbb{R}),$$

where

$$C_+(K, \mathbb{R}) := \{v \in C(K, \mathbb{R}) \,|\, v(y) \geq 0 \ \forall y \in K\}.$$

Then the constraint system of the semi–infinite program (P) may be equivalently written by the *cone constraint*

$$(\, h(x) \, , \, g(x, \cdot) \,) \in C. \tag{1.2}$$

Note that the cone constraint formulation (1.2) fits into the theory of generalized equations handled by Robinson in [43].

By the General Assumptions GA, the function $\sigma : \mathbb{R}^n \to \mathbb{R}^p \times C(K, \mathbb{R})$, defined by

$$\sigma(x) := (h(x), g(x, \cdot))$$

is together with its first derivative $D\sigma(x) = (\nabla h(x), Dg(x, \cdot))$ continuous on $\mathbb{R}^n$, where $\nabla h(x)$ is the $(p \times n)$ – Jacobian matrix of h at x. Further, one has for any $z \in \mathbb{R}^n$,

$$[Dg(x, .)\, z](y) = z^{\mathrm{T}}\, \nabla_x g(x, y) \quad \forall y \in K.$$

Spezializing in these terms Robinson's definition [43] of regularity, we arrive at *Robinson's Constraint Qualification* (RCQ) which is said to hold at x^0 with respect to 1.2, if

$$(0_p, \Theta) \in \text{int}\,(\sigma(x^0) + D\sigma(x^0)\,\mathbb{R}^n - C), \tag{1.3}$$

where "int" means the interior and the bracketed set is the set of all elements $c = \sigma(x^0) + D\sigma(x^0)\,z - (a,b),\ z \in \mathbb{R}^n, (a,b) \in C$. By the results in [42, 44, 59], RCQ is equivalent to

$$D\sigma(x^0)\mathbb{R}^n - \mathbb{R}_+(C - \sigma(x^0)) = Z, \tag{1.4}$$

where $Z := \mathbb{R}^p \times C(K, \mathbb{R})$.

The second version of modelling the constraint system of (P) is to formulate it as a system of smooth equations and one Lipschitzian inequality:

$$\begin{array}{rcl} h_i(x) & = & 0 \qquad (i = 1, ..., p), \\ G(x) & \geq & 0, \\ G(x) & := & \min\limits_{y \in K}\ g(x,y). \end{array} \tag{1.5}$$

Writing down Auslender's CQ [4, 5] for the system (1.4), we obtain another extension of MFCQ: We shall say that EMFCQ* holds at a point $x^0 \in \mathcal{M}$ if the set $\{\nabla h_i(x^0)\}_{i=1,\dots,p}$ is linearly independent as well as either $G(x^0) > 0$, or $G(x^0) = 0$ and

$$\nabla h(x^0)\xi \;=\; 0_p \quad \text{and} \quad \xi^{\mathrm{T}}\, c \;>\; 0\ \forall c \in \partial G(x^0) \tag{1.6}$$

hold for some vector $\xi \in \mathbb{R}^n$, where $\partial G(x^0)$ is Clarke's generalized gradient of G (1.4) at x^0. Of course, the definition of EMFCQ* is justified, since $x \mapsto G(x) := \min_{y \in K} g(x,y)$ is locally Lipschitzian.

It is well-known that the usual one-sided directional derivative $G'(x; z)$ exists for all z and is for $E(x) \neq \emptyset$ (i.e., if $G(x) = 0$) given by the formula

$$G'(x;z) = \min_{y \in E(x)} z^{\mathrm{T}}\, \nabla_x g(x,y). \tag{1.7}$$

Moreover, if $E(x) \neq \emptyset$ then Clarke's generalized gradient is (cf. Rockafellar [50, Prop. 3H,Thm. 4C], [49])

$$\partial G(x) = \text{conv}\,\{\nabla_x\, g(x,y)\,|\, y \in E(x)\}. \tag{1.8}$$

Proposition 2.2 (Equivalent constraint qualifications). *Let x^0 be a point satisfying the constraints (1.1) or, equivalently, (1.2) or (1.4). Then the following conditions are equivalent:*

a. EMFCQ *holds at* x^0.

b. RCQ *holds at* x^0.

c. EMFCQ* *holds at* x^0. ◇

While the equivalence of EMFCQ and EMFCQ* is an immediate consequence of the definitions and the property (1.8), the equivalence of EMFCQ and RCQ was only recently proved by Shapiro [54] for the case that no equations appear (the extension to the case of equations and inequalities is straightforward, see [33]).

2.2 Gauvin's Theorem

Throughout this subsection, we fix $t^0 \in T$ and use the notation (1.9).

Concerning standard (i.e., finite) C^1 programs, Gauvin [14] proved that for a stationary solution x^0, the corresponding set of Langrange multiplier vectors is bounded if and only if the Mangasarian–Fromovitz CQ (MFCQ) holds at x^0. This theorem was extended to optimization problems in Banach spaces (see, e.g., Zowe and Kurcyusz [59] or Penot [42]), where MFCQ has to be replaced by Robinson's CQ. In particular, this result applies to the cone constraint model of the semi–infinite program (P), where by Proposition 2.2, Robinson's CQ may be replaced by EMFCQ. We shall follow a different approach which was proposed by the first author in [34]. This concept also allows a very simple proof of the upper semicontinuity of the stationary point mapping under perturbations, see [34] and Subsection 2.3 below.

In what follows, we prove the Gauvin theorem in the version of [34]. First we note that for a given stationary solution x, by Caratheodory's theorem for cones (cf., e.g., [48, 57]), the second sum in the Kuhn-Tucker condition (1.7) could be restricted to n components (even to $d := \dim \operatorname{cone} \Gamma(x)$ components). However, in view of our extended Gauvin theorem we shall work with $(n+1)$ multipliers associated with inequalities, see Example 2.5.

Let x be any stationary solution of (P) . For a given $r \in \{0, 1, ..., n+1\}$ and a given r-tuple Y in $[E(x)]^r := E(x) \times ... \times E(x)$, with $Y := \emptyset$ when $r = 0$, we set

$$\Lambda^Y(x) := \left\{ (u,v) \in \mathbb{R}^p \times \mathbb{R}_+^{n+1} \,\middle|\, \begin{array}{l} (x, u, v_{[r]}, Y) \text{ satisfies } (1.7)_{t=t^0}, \\ v_j = 0 \quad \forall j : r < j \leq n+1 \end{array} \right\}. \qquad (1.9)$$

The union of the sets $\Lambda^Y(x)$, taken over all $Y \in [E(x)]^r$ and all $r \in \{0,1,...,n+1\}$, will be called the *set of* $(\mathbb{R}^p \times \mathbb{R}^{n+1}_+)$*–multipliers associated with* x and will be denoted by $\Lambda(x)$. By definition, the multiplier set $\Lambda(x)$ associated with a stationary solution x is always nonempty. Of course, even in the case of a finite index set K, the sets of $(\mathbb{R}^p \times \mathbb{R}^{n+1}_+)$–multipliers generally differ from the usual Lagrange multiplier sets.

The following statement and proof are essentially taken from Klatte [34]. Note that the proof of the necessity part even provides a simple bound for the set of $(\mathbb{R}^p \times \mathbb{R}^{n+1}_+)$–multipliers.

Theorem 2.3 (Extended Gauvin theorem) *Let x^0 be a stationary solution of* (P). *Then x^0 satisfies EMFCQ with respect to (1.1) if and only if the set of $(\mathbb{R}^p \times \mathbb{R}^{n+1}_+)$–multipliers $\Lambda(x^0)$ is bounded.* $\diamond$

Proof. For shortness, we set $q(y) := \nabla_x g(x^0, y), y \in K$.

Necessity. Suppose that (1.8) holds at x^0 with some ξ. Let $E(x^0) \neq \emptyset$, otherwise the assertion is trivial. Further, assume $p \geq 1$, for $p = 0$ the proof runs similarly. Since $\Gamma(x^0)$ is compact, there exist $\lambda, \mu > 0$ such that $\xi^{\mathrm{T}} q \geq \lambda$ and $\|H^- q\| \leq \mu$ for all $q \in \Gamma(x^0)$, where H^- is the pseudo-inverse of $\nabla h(x^0)$. Consider any $(u, v) \in \Lambda(x^0)$. Thus there are $r \in \{0, 1, ..., n+1\}$ and $y^1, ..., y^r$ in $E(x^0)$ such that

$$\nabla f(x^0) + \nabla h(x^0) u - \sum_{j=1}^{r} v_j q(y^j) = 0_n \tag{1.10}$$

and $v_j = 0$ for $j > r$. If $r = 0$ then $(u, v) = (-H^- \nabla f(x^0), 0_d)$. If $r \geq 1$ then scalar multiplication by ξ in (1.10), $v \geq 0_d$ and (1.8) imply

$$0 \leq \sum_{j=1}^{n+1} v_j = \sum_{j=1}^{r} v_j \leq \alpha := \lambda^{-1} \xi^{\mathrm{T}} \nabla f(x^0),$$

and so, by (1.8) and (1.10), $\|u\| \leq \beta := \|H^- \nabla f(x^0)\| + \alpha\mu$. Since α and β are independent of the choice of (u, v), $\Lambda(x^0)$ is hence bounded.

Sufficiency. We suppose that x^0 does not satisfy EMFCQ. It will be proved that there is some r-tuple Y in $[E(x^0)]^r$ such that $\Lambda^Y(x^0)$ is unbounded.
Let (u^0, v^0) be any (fixed) element of $\Lambda(x^0)$, i.e., there are $y^1, ..., y^r \in E(x^0)$, $0 \leq r \leq n+1$, such that

$$\nabla f(x^0) + \sum_{i=1}^{p} u_i^0 \nabla h_i(x^0) - \sum_{j=1}^{r} v_j^0 q(y^j) = 0_n. \tag{1.11}$$

If rank $\nabla h(x^0) < p$ then the system

$$\sum_{i=1}^{p} u_i \nabla h_i(x^0) = 0_n$$

has a solution $u^* \neq 0_p$, hence $(u^0 + \theta u^*, v^0) \in \Lambda(x^0)$ for all $\theta \in \mathbb{R}$, and so $\Lambda(x^0)$ is unbounded in this case.
Consider the other case that rank $\nabla h(x^0) = p \geq 1$ or $p = 0$. Since x^0 does not satisfy EMFCQ, we thus conclude from Lemma 2.1 that there are $u^* \in \mathbb{R}^p$, $m \geq 1$, $\bar{y}^1, ..., \bar{y}^m \in E(x^0)$ and $v_1^*, ..., v_m^* > 0$ with

$$\sum_{i=1}^{p} u_i^* \nabla h_i(x^0) - \sum_{l=1}^{m} v_l^* q(\bar{y}^l) = 0_n. \tag{1.12}$$

Now choose any positive real number $\theta > 0$. Combining (1.11) and (1.12), we then obtain

$$\nabla f(x^0) + \sum_{i=1}^{p}(u_i^0 + \theta u_i^*)\nabla h_i(x^0) - \sum_{j=1}^{r} v_j^0 q(y^j) - \theta v_1^* q(\bar{y}^1) - \sum_{l=2}^{m} \theta v_l^* q(\bar{y}^l) = 0_n.$$

Defining $J := \{y^1, ..., y^r, \bar{y}^2, ..., \bar{y}^m\}$ and $W := \text{cone}\,\{q(y) | y \in J\}$, we apply Caratheodory's lemma for cones ([48, Corollary 17.1.2], [57, Thm. 2.2.11]) to the point

$$\nabla f(x^0) - \theta v_1^* q(\bar{y}^1) + \sum_{i=1}^{p}(u_i^0 + \theta u_i^*)\nabla h_i(x^0) \ \in \ W.$$

Since $\dim W \leq n$, there are $N(\theta)$ elements $\tilde{y}_\theta^j$ of J, $0 \leq N(\theta) \leq n$, such that

$$\nabla f(x^0) + \sum_{i=1}^{p}(u_i^0 + \theta u_i^*)\nabla h_i(x^0) - \theta v_1^* q(\bar{y}^1) - \sum_{j=1}^{N(\theta)} v_j(\theta) q(\tilde{y}_\theta^j) = 0_n \tag{1.13}$$

with suitable quantities $v_j(\theta) \in \mathbb{R}_+$. This is true for any $\theta > 0$. Hence, there is an infinite sequence θ_k tending to $+\infty$ as $k \to \infty$ and satisfying $N(\theta_k) \equiv N$, such that (1.13) holds with $\theta = \theta_k$ $(\forall k)$ and with a constant selection $\{\tilde{y}^1, ..., \tilde{y}^N\} \subset J$, $0 \leq N \leq n$. Obviously,

$$(u^0 + \theta_k u^*, \theta_k v_1^*, v_1(\theta_k), ..., v_N(\theta_k), 0_{n-N}) \in \Lambda^Y(x^0)$$

with $Y = [\bar{y}^1, \tilde{y}^1, ..., \tilde{y}^N]$. This defines multiplier vectors for which at least the component $\theta_k v_1^*$ tends to $+\infty$ as $k \to +\infty$. Hence, $\Lambda(x^0)$ is unbounded, which completes the proof by contraposition. $\square$

Remark 2.4 By a suitable modification of the arguments in the proof of Theorem 2.3, one obtains also a proof of the standard Gauvin theorem in the case of a finite index set K. $\diamond$

As mentioned above, the application of the Caratheodory theorem for cones to $(1.7)_{t=t^0}$ suggests to define, instead of (1.9),

$$\tilde{\Lambda}^Y(x) := \left\{ (u,v) \in \mathbb{R}^p \times \mathbb{R}^d_+ \,\middle|\, \begin{array}{l} (x,u,v_{[r]},Y) \text{ satisfies } (1.7)_{t=t^0}\,, \\ v_j = 0 \quad \forall j : r < j \leq d \end{array} \right\},$$

where $d := \dim \operatorname{cone} \Gamma(x) \leq n$. Note that in this setting of $(p+d)$–dimensional multiplier vectors, the necessity part of Theorem 2.3 holds true, but the sufficiency part will fail, in general. This can be illustrated by

Example 2.5 *Consider the linear program in* $\mathbb{R}^2$,

$$\min\{x_1 \mid x_2 \geq 0,\ x_1 - x_2 \geq 0,\ -x_1 - x_2 \geq 0\}.$$

The unique optimal solution (and also stationary solution) is $x^0 = (0,0)$. *Starting from the left, we denote the indices of the constraint by* $j = 1,2,3$. *MFCQ does not hold at* x^0. *Obviously* $E(x^0) = \{1,2,3\}$. *The set* $LM(x^0)$ *of usual Lagrange multipliers is given by all* $v \in \mathbb{R}^3$ *such that*

$$(1,0) = v_1 \cdot (0,1) + v_2 \cdot (1,-1) + v_3 \cdot (-1,-1), \quad v_1, v_2, v_3 \geq 0,$$

hence $LM(x^0) = \{(1,1,0) + \tau \cdot (2,1,1),\ \tau \geq 0\}$. *By definition, the multiplier set* $\Lambda(x^0)$ *is a subset of* $\mathbb{R}^3_+$, *i.e., we have*

$$\Lambda^{[1,2,3]}(x^0) \;=\; LM(x^0).$$

Hence, $\Lambda(x^0)$ *is an unbounded set. This corresponds both to the classical Gauvin theorem and to Theorem 2.3 above. If we would restrict ourselves to multipliers of dimension* $d := \dim \operatorname{cone} \Gamma(x^0)$, *where* $\Gamma(x^0)$ *is defined according to Section 1, we had* $\Gamma(x^0) = \{(0,1),\ (1,-1),\ (-1,-1)\}$, *i.e.,* $d = 2$, *and hence,* $\tilde{\Lambda}^{[1,2]}(x^0) = \{(1,1)\}$ *and* $\tilde{\Lambda}^{[1,3]}(x^0) = \tilde{\Lambda}^{[2,3]}(x^0) = \emptyset$. *This means that the set of "multipliers of minimal representation" associated with the stationary solution* x^0 *is bounded though MFCQ is violated.* $\diamond$

Remark 2.6 In the cone constraint formulation of the semi–infinite program (P),

$$(\tilde{\mathrm{P}}): \qquad \text{minimize} \quad f(x) \quad \text{s.t.} \;\; \sigma(x) \in C,$$

where $\sigma(x) := (h(x), g(x, \cdot))$ and $C := \{0_p\} \times C_+(K, \mathbb{R})$ according to Subsection 2.1, the following version of Gauvin's theorem was given by Zowe and Kurcyusz [59, Theorem 4.1] (see also Penot [42, Corollary 3.7]): If the set of Lagrange multipliers

$$LM(x^0) := \{\lambda \in Z^* \mid \lambda \in C^0,\ \langle \lambda, \sigma(x^0) \rangle = 0,\ \nabla f(x^0) + \lambda \circ D\sigma(x^0) = 0\}$$

is nonempty then the boundedness of $LM(x^0)$ is equivalent to (1.4). Here Z^* denotes the dual space of Z, and C^0 is the polar cone of C, i.e., $C^0 := \{\lambda \in Z^* | \langle \lambda, c \rangle \leq 0\ \forall c \in C\}$. Hence, by Proposition 2.2 and the equivalence of (1.4) and RCQ, the boundedness of $LM(x^0)$ is equivalent to RCQ, EMFCQ and EMFCQ*. Note that one direction of the equivalence, namely, the boundedness of $LM(x^0)$ under RCQ, also follows from [40, 45]. For a relation between the Lagrange multiplier sets $LM(x^0)$ and $\Lambda(x^0)$, we refer to [34, 54]. $\diamond$

2.3 Semicontinuity Results

Let us return to the parametric problem $\mathrm{P}(t)$, $t \in T$. By $S(t)$ we shall denote the set of stationary solutions of $\mathrm{P}(t)$. For $(t, x) \in T \times \mathbb{R}^n$, the union of all sets $\Lambda^Y(t, x)$ with Y in $[E(t, x)]^r$ and $r \in \{0, 1, ..., n+1\}$ will be denoted by $\Lambda(t, x)$, where

$$\Lambda^Y(t, x) := \left\{ (u, v) \in \mathbb{R}^p \times \mathbb{R}_+^{d+1} \,\middle|\, \begin{array}{l} (t, x, u, v_{[r]}, Y) \text{ satisfies (1.7)}, \\ v_j = 0\ \forall j : r < j \leq n+1 \end{array} \right\}. \quad (1.14)$$

This defines a multifunction Λ from $T \times \mathbb{R}^n$ to $\mathbb{R}^p \times \mathbb{R}_+^{n+1}$, where, by construction and by Caratheodory's lemma, we have (see the discussion in the unperturbed case)

$$\Lambda(t, x) \neq \emptyset \quad \Longleftrightarrow \quad x \in S(t), \quad (1.15)$$

i.e., for each stationary solution x of the problem $\mathrm{P}(t)$, $\Lambda(t, x)$ is the associated set of $(\mathbb{R}^p \times \mathbb{R}_+^{n+1})$–multipliers.

The following stability results will extend Robinson's Theorem 2.3 [46] given for finite nonlinear programs in terms of the usual Lagrange multiplier mapping, and they will extend Theorem 2 in [34] given for parametric semi–infinite programs with fixed constraint index set. First we shall prove the closedness and local boundedness of multipliers.

Theorem 2.7 *Suppose that the general assumptions GA are satisfied.*

(a) The multifunction Λ is closed on $T \times \mathbb{R}^n$.

(b) Let x^0 be a stationary solution of $P(t^0)$. If x^0 satisfies EMFCQ with respect to (1.1) at $t = t^0$, then the multifunction Λ is locally bounded at t^0. $\diamond$

Proof. *Auxiliary construction.* In order to unify the proof of (a) and (b), let us consider any point $(\bar{t}, \bar{x})$ in $T \times \mathbb{R}^n$ and arbitrary sequences $\{(t^k, x^k)\}$ in $T \times \mathbb{R}^n$, $\{(u^k, v^k)\}$ in $\mathbb{R}^p \times \mathbb{R}^{n+1}$ with

$$(t^k, x^k) \to (\bar{t}, \bar{x}) \text{ and } (u^k, v^k) \in \Lambda(t^k, x^k) \ (\forall k). \tag{1.16}$$

Hence, by definition of $\Lambda(t^k, x^k)$, one has $x^k \in M(t^k)$ for each k, and

$$0_n = \nabla_x f(t^k, x^k) + \sum_{i=1}^{p} u_i^k \nabla_x h_i(t^k, x^k) - \sum_{j=1}^{r(k)} v_j^k \nabla_x g(t^k, x^k, y^{j\,k}) \tag{1.17}$$

with some $r(k) \in \{0, 1, ..., n+1\}$ and certain points

$$y^{j\,k} \in E(t^k, x^k), \ j = 1, ..., r(k). \tag{1.18}$$

Assumption (1.2), the compactness of $\mathcal{K}$ and the closedness of $t \mapsto K(t)$ on T imply that the intersection of multifunctions

$$(t, x) \mapsto E(t, x) = K(t) \cap \{y \in K | g(t, x, y) = 0\}$$

is closed on T (cf., e.g., [6, Lemma 2.2.3, Thm. 3.1.2]). Further, since K is closed and lower semicontinuous on T, assumption (1.2) and (1.5) imply that the function $G(t, x) := \min_{y \in K(t)} g(t, x, y)$ is continuous at $(\bar{t}, \bar{x})$, hence, together with the continuity of h, we obtain

$$\bar{x} \in M(\bar{t}).$$

Extracting subsequences in (1.18) if necessary, we may assume $r(k) \equiv r$. By the compactness of $\mathcal{K}$, each of the r sequences $\{y^{j\,k}\}$ has a subsequence converging to some element $\bar{y}^j$ which belongs to $E(\bar{t}, \bar{x})$ due to the closedness of the index multifunction $(\cdot, \cdot)$. Hence we may also assume

$$y^{j\,k} \to \bar{y}^j \ , \ \bar{y}^j \in E(\bar{t}, \bar{x}) \quad (j = 1, ..., r). \tag{1.19}$$

(a) *Closedness of* Λ. Given $(\bar{t}, \bar{x}) \in T \times \mathbb{R}^n$, let $\{(t^k, x^k)\}$, $\{(u^k, v^k)\}$ be as in (1.16), and suppose that (u^k, v^k) converges to some $(\bar{u}, \bar{v})$.
Proceeding as in the auxiliary construction, we have $\bar{x} \in M(\bar{t})$, and we may assume that there are some integer $r \in \{0, 1, ..., n+1\}$, sequences $\{y^{j\,k}\}$ and

points $\bar{y}^j$, $j = 1, ..., r$, such that (1.17), (1.18) and (1.19) are satisfied. Thus, by applying (1.4) and by passing to the limit in (1.17), one has

$$0_n = \nabla_x f(\bar{t}, \bar{x}) + \sum_{i=1}^{p} \bar{u}_i \nabla_x h_i(\bar{t}, \bar{x}) - \sum_{j=1}^{r} \bar{v}_j \nabla_x g(\bar{t}, \bar{x}, \bar{y}^j)$$

and $\bar{v}_j = 0$ for $r + 1 \le j \le m + 1$. Thus $(\bar{u}, \bar{v}) \in \Lambda(\bar{t}, \bar{x})$.

(b) *Local boundedness of* Λ. First consider the case $E(t^0, x^0) = \emptyset$. Since $E(\cdot, \cdot)$ is closed at (t^0, x^0) (cf. the argument in the auxiliary construction), $E(t, x)$ is empty for all (t, x) in some neighborhood $\mathcal{W}$ of (t^0, x^0). Hence for $x \in S(t)$ with $(t, x) \in \mathcal{W}$, the condition $g(t, x, y) > 0$ $(\forall y \in K(t))$ will be automatically satisfied and so the Karush–Kuhn–Tucker system (1.7) reduces to

$$h(t, x) = 0_p, \; \nabla_x f(t, x) + \sum_{i=1}^{p} u_i \nabla_x h_i(t, x) = 0_n.$$

Because of the linear independence of the gradients $\nabla_x h_i(t^0, x^0)$, $i = 1, ..., p$, then the desired result is classical (cf. [43]).

Now suppose $E(t^0, x^0) \neq \emptyset$. Recall that EMFCQ holds for the system (1.1) at (t^0, x^0). Then we have to show that there are an open neighborhood $\mathcal{W}$ of (t^0, x^0) and a bounded set $Z \in \mathbb{R}^p \times \mathbb{R}^d_+$ with

$$\Lambda(t, x) \subset Z \quad \forall (t, x) \in \mathcal{W}. \tag{1.20}$$

Assume that, on the contrary, there are sequences $\{(t^k, x^k)\}$, $\{(u^k, v^k)\}$ satisfying

$$(t^k, x^k) \to (t^0, x^0), \; (u^k, v^k) \in \Lambda(t^k, x^k)\, (\forall k) \text{ and } \alpha_k := \|(u^k, v^k)\| \to +\infty.$$

Extracting a subsequence, one may assume that $\alpha_k^{-1}(u^k, v^k)$ converges to some $(\tilde{u}, \tilde{v})$ with $\|(\tilde{u}, \tilde{v})\| = 1$. Reasoning as in the auxiliary construction, we see that, with no loss of generality, there are some r in $\{0, 1, ..., n + 1\}$, sequences $\{y^{jk}\}$ and points $\bar{y}^j \in E(t^0, x^0)$, $j = 1, ..., r$, such that (1.17), (1.18) and (1.19) hold. Dividing by α_k and passing to the limit in (1.17), we find that

$$\begin{aligned} &\sum_{i=1}^{p} \tilde{u}_i \nabla_x h_i(t^0, x^0) - \sum_{j=1}^{r} \tilde{v}_j \nabla_x g(t^0, x^0, \bar{y}^j) \;=\; 0_n, \\ &\tilde{u}_i \in \mathbb{R}\; (i = 1, ..., p), \; \tilde{v}_j \ge 0\; (j = 1, ..., r), \; (\tilde{u}, \tilde{v}) \neq 0_{p+r}, \end{aligned} \tag{1.21}$$

where again (1.4) was used. By Lemma 2.1, (1.21) contradicts EMFCQ for the system (1.1) at (t^0, x^0). Hence, (1.20) is shown. □

The preceding theorem particularly says that the sets of multipliers are closed at all events and compact under EMFCQ. As an immediate consequence, the upper semicontinuity of stationary solutions and multipliers follows.

Theorem 2.8 *Suppose that the general assumptions GA are satisfied. Let x^0 be a stationary solution of P(t^0). If x^0 satisfies EMFCQ with respect to (1.1) at $t = t^0$, then there exist neighborhoods $\mathcal{U}$ of t^0 and $\mathcal{N}$ of x^0 such that the multifunctions Λ and $t \mapsto S(t) \cap \mathcal{N}$ are upper semicontinuous on $\mathcal{U} \times \mathcal{N}$ and $\mathcal{U}$, respectively.* $\diamond$

Proof. Since Z in (1.20) may be supposed to be compact, Theorem 2.7(a) and (1.20) imply that Λ is upper semicontinuous on $\mathcal{W}$, cf., e.g., [6, Lemma 2.2.3].

Let $\mathcal{U}$ and $\mathcal{N}$ be neighborhoods of t^0 and x^0, respectively, such that $\mathcal{U} \times \mathcal{N} \subset \mathcal{W}$. Without loss of generality, one may suppose that $\mathcal{N}$ is compact. Hence, it suffices to show that $t \mapsto S(t) \cap \mathcal{N}$ is closed at each $t \in \mathcal{U}$. Consider an arbitrary sequence $\{(t^k, x^k)\}$ in $\mathcal{W}$ with $t^k \to t \in \mathcal{U}$ and $x^k \in S(t^k) \cap \mathcal{N}\,(\forall k)$. For each k then there is some $(u^k, v^k) \in \Lambda(t^k, x^k) \subset Z$. Since $\{x^k\}$ has an accumulation point $x \in \mathcal{N}$, so, by the upper semicontinuity of Λ, $\{(u^k, v^k)\}$ has an accumulation point $(u, v) \in \Lambda(t, x)$. Hence $x \in S(t)$, by (1.15), which completes the proof. $\square$

It is a trivial fact from linear programming that under the assumptions of Theorem 2 (b), $S(t)$ may become empty for t near t^0. However, if x^0 is a strict local minimizer of P(t^0) under EMFCQ, a slightly perturbed problem P(t) has a local minimizer near x^0; moreover, EMFCQ persists, and so $S(t) \neq \emptyset$ for t close to t^0. This is an immediate consequence of parametric global optimization theory, cf., e.g., [6, Theorems 4.2.1, 4.2.2].

3 METRIC REGULARITY OF THE FEASIBLE SET MAPPING

Section 3 of our paper is devoted to the concept of metric regularity. The close relation between metric regularity of constraint systems and pseudo–Lipschitz continuity of the associated constraint set mapping (for a fundamental study we refer to Mordukhovich [41]) indicates the importance of metric regularity for sensitivity and stability analysis of perturbed optimization problems, see also Section 4 below. Hence, characterizations of metric regularity are of great interest. Robinson [44, 45] has clarified the close connections - in fact, equivalences under suitable assumptions - between constraint qualifications (Mangasarian-Fromovitz CQ, Slater CQ, Robinson CQ etc.) and metric regularity in the classical settings of nonlinear programming problems. Among the large number of publications handling metric regularity of multifunctions on different levels of generality we also mention those of [2, 5, 8, 11, 27, 21, 42, 51, 59].

In the semi–infinite setting, we shall prove that metric regularity is equivalent to EMFCQ, and if no equations appear in (1.1), we shall give a characterization of metric regularity via an explicit growth condition of the function which defines the aggregated inequality constraint.

3.1 Basic Concepts

In the first subsection we define metric regularity and apply Cominetti's basic lemma on the equivalence of metric regularity under right–hand side perturbations and metric regularity under arbitrary perturbations to our setting.

Consider the following parametric system of equations and inequalities, where additionally to (1.1) right–hand side perturbations are included:

$$h_i(x,t) = a_i \quad (i=1,\ldots,p) \quad \text{and} \quad g(t,x,y) \geq b(y) \quad (\forall y \in K(t)), \qquad (1.1)$$

where $t \in T, a = (a_1,\ldots,a_p) \in \mathbb{R}^p$ and $b := b(\cdot) \in C(K,\mathbb{R})$ are viewed as parameters, and K , T , h , and g are given according to Section 1. For

$$\omega := (t,a,b) \in \Omega := T \times \mathbb{R}^p \times C(K,\mathbb{R}),$$

$M(\omega)$ denotes the solution set of the system (1.1).

Specializing the notions of metric regularity in [27, Def. 3.1], [11, Def. 2.1, parametric version] to our situation, we arrive at the following definition.

Let $\omega^0 = (t^0, 0_p, \Theta)$ and $x^0 \in M(\omega^0)$. We shall call the system (1.1) *metrically regular* at (ω^0, x^0) if there exist a neighborhood $\mathcal{U}$ of (ω^0, x^0) and a real number $\beta > 0$ such that for all $(\omega, x) \in \mathcal{U}$ with $\omega = (t,a,b)$,

$$\operatorname{dist}(x, M(\omega)) \leq \beta \cdot \max\{||a - h(t,x)||_\infty, \max_{y \in K(t)} (b(y) - g(t,x,y))_+\}, \qquad (1.2)$$

where $\alpha_+ := \max\{\alpha, 0\}$. Note that (1.2) includes that $M(\omega)$ is nonempty for all ω in some neighborhood of ω^0. When fixing $t = t^0$ in this definition, we shall call the system $(1.1)_{t=t^0}$ *metrically regular* at $(0_p, \Theta, x^0)$ *with respect to right-hand side (RHS) perturbations* (briefly *metrically RHS–regular*). When fixing $t = t^0$ in (1.2) and allowing right–hand side perturbations only of the form $(a, \varepsilon(\cdot))$ with $\varepsilon(y) \equiv \varepsilon$, ε a positive scalar, then we shall say that the system

$(1.1)_{t=t^0}$ is *metrically regular* at $(0_p, \Theta, x^0)$ *with respect to uniform right-hand side (URHS) perturbations* (briefly *metrically URHS–regular*). Clearly, one has metric regularity $\Longrightarrow$ metric RHS–regularity $\Longrightarrow$ metric URHS–regularity.

Cominetti [11, Thm. 2.1] has shown that under mild conditions, metric regularity in its general form and metric RHS–regularity are equivalent. Now we present this result in a version which fits in our context.

Lemma 3.1 *Suppose GA, and let (t^0, x^0) be some element of $T \times \mathbb{R}^n$. If the system (1.1)$_{t=t^0}$ is metrically regular with respect to RHS perturbations at $(0_p, \Theta, x^0)$, then the system (1.1) is metrically regular at $(t^0, 0_p, \Theta, x^0)$.* $\diamond$

Proof. Define for $(t,x) \in T \times \mathbb{R}^n$, $G(t,x) := \min_{y \in K(t)} g(t,x,y)$ and $\tilde{G}(x) := \min_{y \in K(t^0)} g(t^0, x, y)$. In order to apply Theorem 2.1 in Cominetti [11], one has to show that there are neighborhoods $\mathcal{U}_t$ of t^0 and $\mathcal{U}_x := B(x^0, \varepsilon)$ of x^0 such that

a. $\tilde{G}$ and G are continuous on $\mathcal{U}_x$ and $\mathcal{U}_t \times \mathcal{U}_x$, respectively,

b. for each $t \in \mathcal{U}_t$, $G(t,\cdot) - \tilde{G}(\cdot)$ is Lipschitz on $\mathcal{U}_x$ with modulus β^{-1},
where $\beta > 0$ is the modulus of metric RHS–regularity of $(1.1)_{t=t^0}$. By the general assumptions GA and the continuity of $t \mapsto K(t)$, Property a. follows from [6, Chapter 4]. Since g is continuously differentiable with respect to x with continuous derivatives $\nabla_x g(\cdot,\cdot,\cdot)$, there exist for any $\bar{y} \in K(t^0)$ neighborhoods $\mathcal{U}_t^{(\bar{y})}$ of t^0, $\mathcal{U}_x^{(\bar{y})}$ of x^0 and $\mathcal{U}_y^{(\bar{y})}$ of $\bar{y}$ such that

$$\| \nabla_x g(t,x,y) - \nabla_x g(t,z,y) \| \ \le \ \frac{1}{2}\beta^{-1} \tag{1.3}$$

holds for all $(t,x,y) \in \mathcal{U}_t^{(\bar{y})} \times \mathcal{U}_x^{(\bar{y})} \times \mathcal{U}_y^{(\bar{y})}$. Hence, by standard arguments, the compactness of $K(t^0)$ and the continuity of K yield that (1.3) holds for all (t,x,y) with $t \in \mathcal{U}_t$, $x \in B(x^0,\varepsilon)$ and $y \in K(t)$, where $\mathcal{U}_t$ is some neighborhood of t^0 and ε is some positive real number. Hence, for all (t,y) with $t \in \mathcal{U}_t$ and $y \in K(t)$, the functions $g(t,\cdot,y)$ are Lipschitz on $B(x^0,\varepsilon)$ with uniform modulus $\frac{1}{2}\beta^{-1}$. Then, we have for all $t \in \mathcal{U}_t$ and all $x, z \in B(x^0, \varepsilon)$,

$$\begin{aligned}
&G(t,x) - \tilde{G}(x) - G(t,z) + \tilde{G}(z) \\
&\le \min_{y \in K(t)} g(t,x,y) - \min_{y \in K(t^0)} g(t^0,x,y) \\
&\quad - \min_{y \in K(t)} g(t,z,y) + \min_{y \in K(t^0)} g(t^0,z,y)
\end{aligned}$$

$$\leq g(t, x, y_{t,z}) - g(t, z, y_{t,z}) - g(t^0, x, y_{0,x}) + g(t^0, z, y_{0,x})$$
$$\leq \beta^{-1}\|x - z\|,$$

where $y_{t,z}$ solves $\min_{y \in K(t)} g(t, z, y)$, while $y_{0,x}$ solves $\min_{y \in K(t^0)} g(t^0, x, y)$. By changing the role of x and z, hence, b. is shown. This completes the proof of the lemma. □

3.2 Epigraph Representation

In this subsection, we shall present an auxiliary result which will be helpful in proving that EMFCQ implies metric regularity under right–hand side perturbations. This is a slightly strengthened version of Lemma 1 in [16]. The idea of our approach relies essentially on the following observation for a non-parametric inequality system: If one supposes that t^0 is fixed, no equations appear, and for each $y \in K(t^0)$, $g(t^0, x, y)$ has the form $x_n - \tilde{g}(x_1, ..., x_{n-1}, y)$, then the distance between some point and the feasible set can be trivially estimated along the x_n-axis. In the case of the general system (1.1), the estimate (1.2) reduces to a similarly simple form if a parametric epigraph-representation exists. As the subsequent lemma will show, EMFCQ provides this after some local coordinate transformation.

Since, the following lemma concerns P(t) for fixed $t = t^0$, we use again the notation (1.9). Again, let $M(a, b) := M(t^0, a, b)$.

Lemma 3.2 *Assume GA, and assume* EMFCQ *to hold at a point $x^0 \in M(0_p, \Theta)$ with $E(x^0) \neq \emptyset$. Then there exists an open neighborhood $\mathcal{U}_a \times \mathcal{U}_b \times \mathcal{U}_x \times \mathcal{U}_z$ of $(0_p, \Theta, x^0, 0_n)$, an open set $\mathcal{O} \supseteq E(x^0)$ and a local C^1-coordinate transformation $\psi : U_x \to U_z$ with $\psi(x^0) = 0_n$ such that for all $(a, b, x, z) \in \mathcal{U}_a \times \mathcal{U}_b \times \mathcal{U}_x \times \mathcal{U}_z$, $z = \psi(x)$, the following systems are equivalent:*

$$\left\{ \begin{array}{rcl} h(x) & = & a \\ g(x, y) & \geq & b(y) \ \forall y \in K \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{rcl} z_{[p]} & = & a \\ z_n & \geq & \phi(b, z_{[n-1]}, y) \ \forall y \in K^* \end{array} \right\}$$

Here, K^* is a compact set with $E(x^0) \subseteq K^* \subseteq K \cap \mathcal{O}$ and $\phi : \mathcal{U}_b \times (\mathcal{U}_z)_{[n-1]} \times \mathcal{O} \to \mathbb{R}$ is continuous on its domain, differentiable with respect to $(b, z_{[n-1]})$ for every $y \in \mathcal{O}$, and the derivative $(b, z_{[n-1]}, y) \mapsto D_{(b, z_{[n-1]})}\phi(b, z_{[n-1]}, y)$ is continuous on $\mathcal{U}_b \times \mathcal{U}_z \times \mathcal{O}$. Further, a positive real number α exists such that $\phi(b, z_{[n-1]}, y) - z_n \leq \alpha \cdot (b(y) - g(x, y))_+$ holds for all $(b, x, z, y) \in U_b \times U_x \times U_z \times K^*$ which satisfy $z = \psi(x)$. ◇

Proof. Since the proof is very similar to the arguments in proving Lemma 2.1 in Henrion [15], we only sketch the proof of the lemma. Using the idea from the proof of Lemma 2.1 in [15], where a nonparametric system was considered under EMFCQ, one can construct a local diffeomorphism ψ between some open neighborhoods $\mathcal{V}_x$ of x^0 and $\mathcal{V}_z$ of 0_n with the following essential features:

a) $z_{[p]} = h(x)$ (where $z = \psi(x)$) and

b) the n^{th} column of $\nabla\psi^{-1}(0_n)$ equals some EMFCQ vector $\xi \in \mathbf{R}^n$,
this is a vector satisfying (1.8) in the definition of EMFCQ. "∇" refers to the Jacobian. Then, equivalence of the equation parts formulated above is evident. We omit the details.

Concerning the remainder let $y' \in E(x^0)$ be an arbitrary active index at x^0. Define $G_1(b, z, y) := g(\psi^{-1}(z), y) - b(y)$ leading to

$$G_1(\Theta, 0_n, y') = 0 \text{ and } \frac{\partial G_1}{\partial z_n}(\Theta, 0_n, y') = \nabla_x g(x^0, y') \cdot \xi > 0. \tag{1.4}$$

By the general assumptions GA and the construction of ψ, there is some neighborhood $\mathcal{N}^0$ of $(\Theta, 0_n, y')$ such that G_1 is continuous on $\mathcal{N}^0$ and differentiable with respect to (b, z), where the derivative $D_{(b,z)}G_1(\cdot,\cdot,\cdot)$ is continuous on $\mathcal{N}^0$. Hence, there exist neighborhoods $\mathcal{N}_b$ of Θ, $\mathcal{N}_z$ of 0_n and $\mathcal{N}_y$ of y' and exactly one map $\phi : \mathcal{N} := \mathcal{N}_b \times (\mathcal{N}_z)_{[n-1]} \times \mathcal{N}_y \to (\mathcal{N}_z)_n$ such that $\phi(\Theta, 0_{[n-1]}, y') = 0$ and $G_1(b, z_{[n-1]}, \phi(b, z_{[n-1]}, y), y) = 0$ on $\mathcal{N}$, and this map ϕ is continuous on $\mathcal{N}$ (cf., e.g., Deimling [12, Thm. 15.1]). Without loss of generality, assume the neighborhoods being so small that each (b, z, y) in $\mathcal{N}_b \times \mathcal{N}_z \times \mathcal{N}_y$ fulfils

$$\frac{\partial G_1}{\partial z_n}(b, z, y) = \nabla_x g(x, y) \cdot \eta > 0,$$

where $x = \psi^{-1}(z)$ and η is the n–th column of $\nabla\psi^{-1}(z)$. The latter was possible by continuity of ∇g and (1.4). Thus, for each $y \in \mathcal{N}_y$, the standard implicit function theorem for C^1 functions (see [12, Corollary 15.1]) may be applied to $G_1(\cdot,\cdot,y)$, and it yields that $\phi(\cdot,\cdot,y)$ is continuously differentiable on a neighborhood of $(\Theta, 0_{n-1})$, which is (possibly strongly) contained in $\mathcal{N}_b \times (\mathcal{N}_z)_{[n-1]}$. Arguing as in the proofs of Corollary 15.1 (Implicit Function Theorem) and Theorem 15.2 (Inverse Function Theorem) in [12], one easily may show, that the continuity of $D_{(b,z)}G_1(\cdot,\cdot,\cdot)$ near $(\Theta, 0_n, y')$ carries over to the continuity of $D_{(b,z_{[n-1]})}\phi(\cdot,\cdot,\cdot)$ with respect to some neighborhood of $(\Theta, 0_{n-1}, y')$.

For all (b, z, y) sufficiently near to $(\Theta, 0_n, y')$, we thus have

$$G_1(b,z,y)\begin{cases}>0\\=0\end{cases} \Leftrightarrow z_n - \phi(b, z_{[n-1]}, y)\begin{cases}>0\\=0\end{cases}. \tag{1.5}$$

Next, we define the function

$$G_2(b,z,y) := G_1(b,z,y) + \gamma\cdot(\phi(b, z_{[n-1]}, y) - z_n),$$

where

$$\gamma := \frac{1}{2}\cdot \min_{y\in E(x^0)} \frac{\partial G_1}{\partial z_n}(\theta, 0_n, y) > 0,$$

hence

$$G_2(\Theta, y', 0_n) = 0 \text{ and } \frac{\partial G_2}{\partial z_n}(\Theta, y', 0_n) > 0.$$

Again, the implicit function theorems apply, and, due to uniqueness of the resolving function and to $G_2(b, z_{[n-1]}, \phi(b, z_{[n-1]}, y), y) = 0$, one gets locally:

$$G_2(b,z,y)\begin{cases}>0\\=0\end{cases} \Leftrightarrow z_n - \phi(b, z_{[n-1]}, y)\begin{cases}>0\\=0\end{cases}. \tag{1.6}$$

Similar equivalences like (1.5) and (1.6) may be established around all active indices $y' \in E(x^0)$. Due to compactness of $E(x^0)$ there exist a neighborhood $\mathcal{U}_b$ of Θ, a neighborhood $\mathcal{U}_z$ of 0_n, an open set $\mathcal{U}_y \supset E(x^0)$ as well as a real–valued function ϕ defined on $\mathcal{U}_b \times (\mathcal{U}_z)_{[n-1]} \times \mathcal{U}_y$ with the smoothness properties as above such that

$$\begin{aligned} G_1(b,z,y)\begin{cases}>0\\=0\end{cases} &\Leftrightarrow\ z_n - \phi(b, z_{[n-1]}, y)\begin{cases}>0\\=0\end{cases} \\ &\Leftrightarrow\ G_1(b,z,y) + \gamma\cdot(\phi(b, z_{[n-1]}, y) - z_n)\begin{cases}>0\\=0\end{cases}\end{aligned} \tag{1.7}$$

holds for all variables in the indicated neighborhoods. Since one may find an arbitrarily small open set $\mathcal{O}$, $E(x^0) \subset \mathcal{O} \subset \operatorname{cl}\mathcal{O} \subset \mathcal{U}_y$, such that all indices $y \in K \setminus \mathcal{O}$ remain inactive for all $x \in \mathcal{U}_x$ (with $\mathcal{U}_x$ sufficiently small), the inequality part in the statement of the lemma is proved by the first equivalence in (1.7), one may take $K^* := K \cap \operatorname{cl}(\mathcal{O})$).

Concerning the last statement of the lemma, let $(b, x, z, y) \in \mathcal{U}_b \times U_x \times U_z \times K^*$ be arbitrary with $z = \psi(x)$. If $\phi(b, z_{[n-1]}, y) - z_n \le 0$, then the statement holds trivially. Let $\phi(b, z_{[n-1]}, y) - z_n > 0$ then the second equivalence in (1.7) gives:

$$\begin{aligned} 0 < \phi(b, z_{[n-1]}, y) - z_n &< (1/\gamma)\cdot(-G_1(b,z,y)) \\ &= (1/\gamma)\cdot(b(y) - g(x,y)) \\ &= (1/\gamma)\cdot(b(y) - g(x,y))_+. \end{aligned}$$

This completes the proof with $\alpha = 1/\gamma$. $\square$

3.3 Metric Regularity and Constraint Qualifications

Now we are going to prove the main results of Section 3. Again t^0 is fixed, we use the notation (1.9) and $M(a,b) := M(t^0, a, b)$.

Theorem 3.3 *Consider the system (1.1) and suppose the general assumptions GA. Let $\omega^0 := (t^0, 0_p, \Theta)$. If* EMFCQ *holds at $x^0 \in M(\omega^0)$, then system (1.1) is metrically regular at (ω^0, x^0).* $\diamond$

Proof. By Lemma 3.1 it suffices to show that (1.1) is metrically RHS–regular at $(0_p, \Theta, x^0)$.
In the case $E(x^0) = \emptyset$, Assumption (1.2) yields that in some neighborhood of $(0_p, \Theta, x^0)$, system $(1.1)_{t=t^0}$ is exclusively described by the equations. But then one deals with classical MFCQ in finite optimization and the stated proposition follows from [44]. For $E(x^0) \neq \emptyset$ Lemma 3.2 may be applied. We consider the system

$$z_{[p]} = u \quad \text{and} \quad z_n \geq \phi(b, z_{[n-1]}, y)) \; \forall y \in K^* \tag{1.8}$$

in the setting of Lemma 3.2. In order to verify metric regularity, let $(a, b, \bar{x}, \bar{z})$ in $\mathcal{U}_a \times \mathcal{U}_b \times \mathcal{U}_x \times \mathcal{U}_z$ with $\bar{z} = \psi(\bar{x})$ be arbitrarily chosen. Define $M^*(a,b) := \{z \in \mathcal{U}_z \mid z \text{ satisfies } (1.8)\}$. Clearly $M^*(a,b) = \psi(\mathcal{U}_x \cap M(a,b))$, by Lemma 3.2. Define a reference point $z^* \in M^*(a,b)$ by

$$z^*_{[n-1]} := (a, \bar{z}_{p+1}, \ldots, \bar{z}_{n-1})$$

and

$$z^*_n := \max\{\bar{z}_n, \phi^*(b, (z^*)_{[n-1]})\}$$

where $\phi^*(b, z_{[n-1]}) := \max_{y \in K^*} \phi(b, z_{[n-1]}, y)$. Feasibility of z^* follows from Lemma 3.2. Then

$$\begin{aligned}
&\operatorname{dist}(\bar{z}, M^*(a,b)) \\
&\leq c \cdot \| \bar{z} - z^* \|_\infty \\
&= c \cdot \max\{\|\bar{z}_{[p]} - a\|_\infty, |\bar{z}_n - z^*_n|\} \\
&= \begin{cases} c \cdot \|\bar{z}_{[p]} - a\|_\infty, & \text{if } \bar{z}_n = z^*_n \\ c \cdot \max\{\|\bar{z}_{[p]} - a\|_\infty, \phi^*(b, (z^*)_{[n-1]}) - \bar{z}_n\} & \text{if } \bar{z}_n < z^*_n \end{cases},
\end{aligned} \tag{1.9}$$

where c is some positive factor of norm equivalence.
Recall that ψ^{-1} is continuously differentiable on $\mathcal{U}_z$ and may be hence assumed, without loss of generality, being Lipschitzian there with some modulus $\varepsilon > 0$. ¿From (1.8) one concludes (with $\bar{z} = \psi^{-1}(\bar{x})$):

$$\begin{aligned}
&\operatorname{dist}(\bar{x}, M(a,b)) \\
&\le \operatorname{dist}(\bar{x}, \mathcal{U}_x \cap M(a,b)) \\
&= \inf_{z \in M^*(a,b)} \| \psi^{-1}(\bar{z}) - \psi^{-1}(z) \| \\
&\le \varepsilon \cdot \operatorname{dist}(\bar{z}, M^*(a,b)) \\
&\le \varepsilon \cdot c \cdot \begin{cases} \|\bar{z}_{[p]} - a\|_\infty, & \text{if } \bar{z}_n = z_n^* \\ \max\{\|\bar{z}_{[p]} - a\|_\infty, \phi^*(b, (z^*)_{[n-1]}) - \bar{z}_n\} & \text{if } \bar{z}_n < z_n^* \end{cases}
\end{aligned} \tag{1.10}$$

Note that the functions $\phi(\cdot,\cdot,y)$, $y \in K^*$, are differentiable on some neighborhood of $(\Theta, \bar{z})$ with continuous derivatives $D_{(v,z)}\phi(\cdot,\cdot,\cdot)$, and so, the Lipschitz continuity arguments in the proof of Lemma 3.1 apply to the maximum function ϕ^* . Thus, without loss of generality, ϕ^* may be assumed being Lipschitzian on $\mathcal{U}_v \times [\mathcal{U}_z]_{n-1}$ with some modulus $\delta > 0$. Taking account of the second assertion in Lemma 3.2 we thus obtain in the case $\bar{z}_n < z_n^*$ that for some $y^* \in \operatorname{argmax}_{y \in K^*} \phi(b, \bar{z}_{[n-1]}, y)$,

$$\begin{aligned}
&\phi^*(b, (z^*)_{[n-1]}) - \bar{z}_n \\
&\le \ \phi^*(b, (z^*)_{[n-1]}) - \phi^*(b, \bar{z}_{[n-1]}) + \phi^*(b, \bar{z}_{[n-1]}) - \bar{z}_n \\
&\le \ \delta \cdot \| a - \bar{z}_{[p]} \|_\infty + \phi(b, \bar{z}_{[n-1]}, y^*) - \bar{z}_n \\
&\le \ \delta \cdot \| a - \bar{z}_{[p]} \|_\infty + \alpha \cdot (b(y^*) - g(\bar{x}, y^*))_+ \\
&\le \delta \cdot \| a - \bar{z}_{[p]} \|_\infty + \alpha \cdot \max_{y \in K} (b(y) - g(\bar{x}, y))_+ .
\end{aligned}$$

Combining this with (1.9), we have

$$\begin{aligned}
&\operatorname{dist}(\bar{x}, M(a,b)) \\
&\le \ c \cdot \varepsilon \cdot \max\{ \| a - \bar{z}_{[p]} \|_\infty ,\ \delta \cdot \| a - \bar{z}_{[p]} \|_\infty + \alpha \cdot \max_{y \in K} (b(y) - g(\bar{x}, y))_+ \}.
\end{aligned}$$

By taking $\bar{z}_{[p]} = h(\bar{x})$ into account, this yields the required condition of metric regularity with modulus $\beta := c \cdot \varepsilon \cdot \max\{1, \alpha + \delta\} > 0$. □

Theorem 3.4 *If the system (1.1) is metrically regular at $x^0 \in M(\omega^0)$ with respect to uniform right–hand side perturbations, then* EMFCQ *holds at x^0.* ◇

Proof. For $\varepsilon \in \mathbb{R}$ let $\hat{\varepsilon}$ be the function defined by $\hat{\varepsilon}(y) = \varepsilon$ ($\forall y \in K$). By hypothesis, there exist neighborhoods $\mathcal{U}$ of 0_p, $\mathcal{V}$ of x^0 and positive real numbers

β, γ such that for all $(a,x) \in \mathcal{U} \times \mathcal{V}$ and all $\varepsilon \in [0,\gamma]$,

$$\mathrm{dist}\,(x, M(a, -\hat{\varepsilon})) \leq \beta \cdot \max\{||(a - h(x)||_\infty\,,\ \max_{y \in K}(-\varepsilon - g(x,y))_+\}$$

The compactness of K and the continuity of g entail that for some neighborhood $\mathcal{V}' \subseteq \mathcal{V}$ of x^0, $g(x,y) \geq -\gamma$ $(\forall (a,x) \in \mathcal{U} \times \mathcal{V}')$. Hence, we have for the solution set mapping $a \to \widetilde{M}(a)$ of the system $h(x) = a$ that if $(a,x) \in \mathcal{U} \times \mathcal{V}'$ then

$$\begin{aligned} \mathrm{dist}\,(x, \widetilde{M}(a)) &\leq dist(x, M(a, -\hat{\gamma})) \\ &\leq \beta \cdot \max\{||a - h(x)||_\infty\,,\ \max_{y \in K}(-\gamma - g(x,y))_+\} \\ &= \beta \cdot ||a - h(x)||_\infty, \end{aligned}$$

which implies by classical regularity theory (cf., e.g., Robinson [44, Cor. 3]) that the linear independence requirement of EMFCQ is satisfied.
Now we shall prove the inequality condition in EMFCQ. If $E(x^0) = \emptyset$ then locally one deals with equality constraints only, but then the assertion follows from [44] again. Therefore let $E(x^0) \neq \emptyset$. Then $\min_{y \in K} g(x^0, y) = 0$. Consider a sequence $\varepsilon_k \downarrow 0$ with $\varepsilon_k > 0$. Since $(a,b) = (0_p, \widehat{\varepsilon_k})$ are feasible right-hand side perturbations of (1.1), metric regularity implies that to each ε_k there belongs a point $x^k \in M(0_p, \widehat{\varepsilon_k})$ fulfilling

$$||\ x^k - x^0\ || = \mathrm{dist}\,(x^0, M(0_p, \widehat{\varepsilon_k})) \leq \beta \cdot \max_{y \in K}(\varepsilon_k - g(x^0, y))_+ = \beta \cdot \varepsilon_k. \quad (1.11)$$

For all $y \in E(x^0)$ one has $g(x^0, y) = 0$ and, due to feasibility of x^k, one has $g(x^k, y) \geq \varepsilon_k$. Taking account of (1.11), this leads to $(\forall y \in E(x^0))$:

$$\begin{aligned} \frac{1}{\beta} &\leq \frac{\varepsilon_k}{||\ x^k - x^0\ ||} \leq \frac{g(x^k, y) - g(x^0, y)}{||\ x^k - x^0\ ||} \\ &= \frac{\nabla_x g(x^0, y)(x^k - x^0) + o(x^k - x^0)}{||\ x^k - x^0\ ||} \end{aligned} \quad (1.12)$$

By (1.11) it holds $\lim_{k \to \infty} ||\ x^k - x^0\ || = 0$. Without loss of generality we may assume $\lim_{k \to \infty} ||x^k - x^0||^{-1}(x^k - x^0) = \xi \in \mathbb{R}^n$. Transition to the limits on both sides of (1.12) provides $\nabla_x g(x^0, y) \cdot \xi \geq (1/\beta) > 0\,(\forall y \in E(x^0))$. On the other hand, since $x^0 \in M(0_p, \Theta)$ and $x^k \in M(0_p, \widehat{\varepsilon_k})$, it follows that $h(x^0) = h(x^k) = 0_p$, hence $\nabla h_i(x^0)(x^k - x^0) = o(x^k - x^0)$. Dividing by $||\ x^k - x^0\ ||$ and passing to the limits yield $\nabla h_i(x^0) \cdot \xi = 0\,(i = 1, \ldots, p)$ □

Remark 3.5 By Theorem 3.3 and Theorem 3.4 we may conclude that for parametric systems of the type (1.1) and under the general assumptions GA, the concepts of metric regularity, metric RHS–regularity and metric URHS–regularity are equivalent each to each other. Moreover, metric regularity is equivalent to the constraint qualification EMFCQ. This fact could be alternatively proved by applying results in the cone constraint form of (1.1): Taking the equivalence of EMFCQ and Robinson's CQ (Shapiro [54]) in the semi–infinite setting into account, we obtain from Robinson [43, Thm. 1] that EMFCQ (or, equivalently, RCQ) implies metric regularity. The converse is also true, this follows from Borwein [8, Thms. 6.1, 6.4] or Cominetti [11, Corollary 2.1]. Further, note that in the case of writing (1.1) in the parametric form of (1.4), the implication EMFCQ* $\Rightarrow$ metric RHS–regularity is a special case of Auslender [5, Thm. 1.1] (combined with [4, Thm. 2.1]. This result can be also found in Levitin [39]. $\diamond$

3.4 Explicit Growth of Constraint Function

Usually, characterizations of metric regularity for constraint mappings are formulated as constraint qualifications in terms of subgradients or gradients of the functions involved (compare conditions EMFCQ, EMFCQ* and RCQ of the previous sections). Sometimes however, explicit conditions on functional values might be easier to check, as it was found, for instance, in the context of verifying metric regularity of chance constraints (see Henrion and Römisch [17]). In the following, we shall establish a growth condition for a continuous constraint function $F : \mathbb{R}^n \to \mathbb{R}$ describing a single inequality

$$F(x) \ \geq \ 0 \tag{1.13}$$

and then apply it to the setting of semi-infinite programming. We make use of Mordukhovich's [41] complete characterization of metric regularity for multifunctions between finite dimensional spaces. First note that, in consistency with the previous definition, the constraint $F(x) \geq 0$ is said to be metrically regular with respect to right-hand side perturbations at some feasible x^0 if there exists some $\beta > 0$ and a neighborhood $\mathcal{U}$ of $(0, x^0)$ such that

$$\mathrm{dist}\ (x, M(\varepsilon)) \leq \beta \cdot (\varepsilon - F(x))_+ \ \forall (\varepsilon, x) \in \mathcal{U},$$

where $M(\varepsilon) = \{x \in \mathbb{R}^n \mid F(x) \geq \varepsilon\}$. Denote by ∂_a the approximate subdifferential, which for general lower semicontinuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\partial_a \phi(z) = \limsup_{\substack{x \to z \\ \phi(x) \to \phi(z)}} \partial^- \phi(x),$$

where 'limsup' denotes the limit superior of multifunctions in the Kuratowski-Painlevé sense and

$$\partial^- \phi(x) = \{x^* \in \mathbb{R}^n \mid \langle x^*, h\rangle \le d^-\phi(x;h)\ \forall h \in \mathbb{R}^n\}$$

refers to the Dini subdifferential with lower Dini directional derivative

$$d^-\phi(x;h) = \liminf_{\substack{u\to h\\ t\downarrow 0}} t^{-1}(\phi(x+tu) - \phi(x)).$$

More explicitly, one has

$$x^* \in \partial_a\phi(z) \iff \text{there exist sequences } x^n \to z, (x^n)^* \to x^*, \text{ such that } \phi(x^n) \to \phi(z) \text{ and } (x^n)^* \in \partial^-\phi(x^n)\ (\forall n)$$

For continuous ϕ, the condition $\phi(x^n) \to \phi(z)$ may be omitted of course.

Specializing one of Mordukhovich's [41] characterizations of metric regularity to a single inequality $F(x) \ge 0$, we have

Lemma 3.6 *Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is continuous, and let x^0 satisfy $F(x^0) \ge 0$. Then the following two statements are equivalent:*

1. *$F(x) \ge 0$ is metrically regular at x^0 with respect to right–hand side perturbations.*
2. *$F(x^0) > 0$ or $0 \notin \partial_a(-F)(x^0)$.* ◇

This allows to prove the main result of the present subsection.

Theorem 3.7 *Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous. The constraint $F(x) \ge 0$ is metrically regular at some feasible x^0 with respect to right–hand–side perturbations if and only if $F(x^0) > 0$ (non-binding case) or the following growth condition is fulfilled: There exists some $\rho > 0$ and a neighborhood $\mathcal{U}$ of x^0, such that arbitrarily close to any $x \in \mathcal{U}$ one may find some y with $F(y) > F(x) + \rho\|y - x\|$.* ◇

Proof. ¿From continuity arguments it is obvious that the non-binding case entails metric regularity at x^0 (both sides become zero in the inequality of statement 1 of Lemma 3.6), so let the growth condition be fulfilled at x^0. Consider an arbitrary $x \in \mathcal{U}$. Now, the growth condition provides the existence of a sequence $y^n \to x$ such that $F(y^n) > F(x) + \rho\|y^n - x\|$. In particular, $y^n \neq x$, and, without loss of generality, one may assume that $h^n := \|y^n - x\|^{-1}(y^n - x) \to h$. It follows

$$\begin{aligned} d^-(-F)(x;h) &= \liminf_{\substack{u \to h \\ t \downarrow 0}} t^{-1}(F(x) - F(x+tu)) \\ &\leq \liminf_{n\to\infty} \|y^n - x\|^{-1}(F(x) - F(x + \|y^n - x\| \cdot h^n)) \\ &= \liminf_{n\to\infty} \|y^n - x\|^{-1}(F(x) - F(y^n)) \leq -\rho \end{aligned}$$

For $x^* \in \partial^-(-F)(x)$ one gets $\langle x^*, h\rangle \leq -\rho$, hence $\|x^*\| \geq \rho$ due to $\|h\| = 1$. Since this is true for all $x \in \mathcal{U}$, the explicit definition of the approximate subdifferential, given above, yields $\|x^*\| \geq \rho > 0$ for all $x^* \in \partial_a(-F)(x^0)$. In particular, $0 \notin \partial_a(-F)(x^0)$, so metric regularity follows from the equivalence of statements 1. and 2. of Lemma 3.6.

In order to show the reverse direction of the Theorem, assume metric regularity and the binding case. According to the notation introduced above, there exist $\beta, \eta_1, \eta_2 > 0$ such that (with B^0 denoting open balls):

$$\text{dist } (x, M(\varepsilon)) \leq \beta \cdot (\varepsilon - F(x))_+ \quad \forall (x,\varepsilon) \in B^0(x^0,\eta_1) \times (-\eta_2, \eta_2).$$

By continuity of F, one may choose η_1 small enough to meet

$$|F(x)| < \eta_2 \quad \forall x \in B^0(x^0, \eta_1).$$

Now, consider any $x \in B^0(x^0,\eta_1)$ and any $\delta > 0$. Select some $\alpha > 0$ such that $\alpha \leq \beta^{-1}\delta$ and $|F(x) + \alpha| < \eta_2$. With $\varepsilon := F(x) + \alpha$, the distance estimation above applies to x and ε:

$$\text{dist } (x, M(\varepsilon)) \leq \beta \cdot (\varepsilon - F(x))_+ = \beta\alpha.$$

In particular, the set $M(\varepsilon)$ is nonempty and it is also closed (by continuity of F). Consequently, there exists some $\tilde{x} \in M(\varepsilon)$ with

$$\|x - \tilde{x}\| = \text{dist } (x, M(\varepsilon)) \leq \beta\alpha \leq \delta.$$

Furthermore, $\tilde{x} \in M(\varepsilon)$ implies that $F(\tilde{x}) - F(x) \geq \alpha$, hence

$$\|x - \tilde{x}\| \leq \beta\alpha \leq \beta(F(\tilde{x}) - F(x)).$$

Since x and δ were chosen arbitrarily, this last relation implies the asserted growth condition for F (with $\mathcal{U} := B^0(x^0, \eta_1)$ and $\rho := (2\beta)^{-1}$). □

We note, that more general results in this direction have been obtained in [17]. In particular, a slightly generalized growth condition compared to the one considered here, is sufficient to guarantee metric regularity of a constraint system consisting of a finite number of inequalities which are defined by upper semi-continuous (not just locally Lipschitzian) functions. Moreover, the equivalence stated in Theorem 3.7 also holds true when considering metric regularity with respect to some additional unperturbed subset $C \subseteq \mathbb{R}^n$.

As an application of Theorem 3.7 in the context of semi-infinite programming, we have

Corollary 3.8 *Under the general assumptions GA, and with the notations of subsection 3.3, the constraint system (1.1) without equalities is metrically regular at some* $(t^0, 0_p, \Theta, x^0)$ *with some feasible* x^0*, if and only if* $E(x^0) = \emptyset$ *(non-binding case) or the following growth condition is fulfilled: There exists some* $\rho > 0$ *and a neighborhood* $\mathcal{U}$ *of* x^0*, such that arbitrarily close to any* $x \in \mathcal{U}$ *one may find some* x' *with*

$$g(x', y) > g(x, \bar{y}) + \rho \|x - x'\| \text{ for all } y \in K \text{ and for some } \bar{y} \in K.$$

$\diamond$

Proof. Using the minimum function $G(x) = \min\{g(x, y) \mid y \in K\}$, we see that $E(x^0) = \emptyset$ is equivalent to $G(x^0) > 0$ and the growth condition of the corollary is equivalent to the growth condition of Theorem 3.7 with F replaced by G. Then, the theorem provides the assertion of the corollary, taking into account the equivalences (cf. Remark 3.5)

$$\begin{aligned} \text{metric regularity of } (1.1) &\iff \text{URHS- regularity of } (1.1) \\ &\iff \text{metric regularity of } G(x) \geq 0 \text{ with respect} \\ &\qquad \text{to right-hand side perturbations.} \end{aligned}$$

$\square$

4 STABILITY OF LOCAL MINIMIZERS

This section is devoted to the stability behavior of local minimizers of the parametric problem introduced in Section 1, as far as the stability properties follow from metric constraint regularity and known abstract stability results

[1, 5, 6, 29, 32, 47]. The proofs will be only sketched. We denote by B_n the closed unit ball in $\mathbb{R}^n$, by $B(z,\varepsilon)$ the closed ε - neighborhood of z both in $\mathbb{R}^n$ and T, and by $\operatorname{cl} Q$ the closure of $Q \subset \mathbb{R}^n$. Given $t^0 \in T$, a strict local minimizer x^0 of $\mathrm{SIP}(t^0)$ is called to be *of order* $\kappa \geq 1$ if there are real numbers $\varrho > 0$ and $c > 0$ such that

$$f(x,t^0) \geq f(x^0,t^0) + c\|x - x^0\|^\kappa \qquad \forall\ x \in M(t^0) \cap B(x^0,\varrho).$$

Denote by $\psi_Q(t)$ the set of all global minimizers for $f(.,t)$ on $M(t) \cap \operatorname{cl} Q$, $t \in T$, $Q \subset \mathbb{R}^n$, and by $\psi_{loc}(t)$ the set of all local minimizers for $f(.,t)$ w.r. to $M(t)$.

Following concepts in [1, 5, 29, 47], we shall say that a local minimizer x^0 of $\mathrm{SIP}(t^0)$ is *stable* (w.r. to $\mathrm{SIP}(t)$, $t \in T$), if for some positive real numbers ε', δ' and for each $\varepsilon \in (0,\varepsilon')$ there is some $\delta \in (0,\delta')$ such that with $Q := B(x^0,\varepsilon')$,

$$\psi_Q(t^0) = \{x^0\}, \tag{1.1}$$
$$\emptyset \neq \psi_Q(t) \subset \psi_{loc}(t) \qquad \forall t \in B(t^0,\delta'), \tag{1.2}$$
$$\psi_Q(t) \subset B(x^0,\varepsilon) \qquad \forall t \in B(t^0,\delta). \tag{1.3}$$

A stable local minimizer is called to be *stable with rate* r, $r \in (0,1]$, if the relation between ε and δ in (1.3) is quantified (with some $\beta > 0$) by

$$\|x - x^0\| \ \leq\ \beta\, d(t,t^0)^r \qquad \forall t \in B(t^0,\delta') \quad \forall x \in \psi_Q(t). \tag{1.4}$$

Obviously (1.1) – (1.3) include that the multivalued selection ψ_Q of the multifunction ψ_{loc} is (upper and lower) continuous at t^0.

For both stability properties, we now recall sufficient conditions which specialize more abstract results to $\mathrm{SIP}(t)$. Note that by the general assumptions, M is a closed multifunction.

Proposition 4.1 *Suppose that GA holds. Let $t^0 \in T$, and let x^0 be a strict local minimizer of* $\mathrm{P}(t^0)$. *If EMFCQ holds at x^0, then x^0 is stable.* ◇

Proof. (Outline) Since EMFCQ at x^0 implies metric regularity, it is not difficult to show that the general assumptions GA imply that M is lower semicontinuous at (t^0,x^0). A direct proof that EMFCQ implies lower semicontinuity, one may find in [26]. Then the assertion is a specialization of Klatte [29, Thm. 1] or Robinson [47, Thm. 4.3]. □

Next we give the quantified version of the previous proposition. A multifunction Ψ from T to $\mathbb{R}^n$ is called *Lipschitzian* on $\tilde{T} \subset T$ if there is some constant $c > 0$

such that $\Psi(t') \subset \Psi(t) + c\|t' - t\| B_n$ for all $t, t' \in \tilde{T}$. Following Aubin [2] and Rockafellar [51], we shall say that Ψ is *pseudo-Lipschitzian at* $(t^0, x^0) \in \operatorname{graph} \Phi$ *with modulus* $c > 0$, if there exist neighborhoods $\mathcal{W}$ of t^0 and $\mathcal{X}$ of x^0 with $\Psi(t') \cap \mathcal{X} \subset \Psi(t) + c\|t' - t\| B_n$ for all $t, t' \in \mathcal{W}$.

Proposition 4.2 *Suppose that GA holds. Let $t^0 \in T$, and let x^0 be a strict local minimizer to* $\mathrm{SIP}(t^0)$ *of order $\kappa \geq 1$. Suppose that for some neighborhoods $\mathcal{U}$ of t^0 and $\mathcal{V}$ of x^0, the functions h and g are Lipschitz continuous on $\mathcal{U} \times \mathcal{V}$ and $\mathcal{U} \times \mathcal{V} \times K$ respectively, and the multifunction K is Lipschitzian on $\mathcal{U}$. Moreover, let for some $\gamma \in (0, 1]$ and $c_f \in (0, +\infty)$*

$$|f(x, t^0) - f(x', t)| \leq c_f \ (\|x - x'\| + d(t, t^0)^\gamma) \qquad \forall t \in \mathcal{U} \quad \forall x, x' \in \mathcal{V}.$$

If x^0 satisfies EMFCQ with respect to (1.1) at $t = t^0$, then x^0 is stable with rate $r = \gamma\kappa^{-1}$. ◇

Proof. (Outline) The principal idea already appears in [1, 5, 30] in the context of standard nonlinear programs. In our context, the assumptions yield that $G(t, x) := \min_{y \in K(t)} g(t, x, y)$, $(t, x) \in T \times \mathcal{V}$ is Lipschitz on $\mathcal{U} \times \mathcal{V}$, see, e.g., [3]. By Theorem 3.3, the Lipschitzian constraint system $(1.1)_{t=t^0}$ is metrically regular, hence M is pseudo–Lipschitzian at t^0, see, e.g., [41]. Now, the assertion follows from [30, Thm. 2] (see also Theorem 2.2 in [32]). □

For second-order sufficient conditions ensuring that x^0 is a strict local minimizer of order 2 to $\mathrm{P}(t^0)$ under twice differentiability of the data with respect to x, we refer, e.g., to Hettich and Jongen [18], Shapiro [53], Hettich and Still [19], Kawasaki [28]. Under the reduction ansatz, conditions for strict local minimality of order $\kappa = 1$ (also called *strong unicity* or *weak sharp minimizer of order 1*) can be found, e.g., in Hettich and Zencke [20] (see also [7]).

Proposition 4.2 provides (for $\gamma = 1$) Hölder stability of order κ^{-1} for strict local minimizers of order κ. Simple examples show that in the case $\kappa = 2$, stability of order 1 (upper Lipschitz continuity) may not be expected [35]. Under certain constraint qualifications and second-order optimality conditions on $\mathrm{P}(t^0)$, Shapiro [54] obtains stability of order 1 for global minimizers of $\mathrm{P}(t)$ for $K(t) \equiv K$.

5 CONCLUDING REMARKS

In the previous sections, we concentrated on certain kinds of regularity and stability studies and shelved many other questions which might be also handled under the title of the paper. Let us mention a few of them.

A series of papers by Jongen, Twilt, Rückmann, Weber and others is devoted to the structural analysis and the topological stability of the feasible set of semi–infinite C^2–optimization problems, for recent progress in this topic we refer, e.g., to [23, 25, 26, 52].

Essential refinements of Proposition 4.2 may be obtained when the data are twice differentiable. The use of second–order information allows to derive (directional) differential stability as well as (upper) Hölder or Lipschitz stability properties of optimal values and optimal solutions to perturbed optimization problems in Banach spaces. These results are applicable for perturbed semi–infinite programs by using the cone constraint setting. An excellent survey has been given recently by Bonnans and Shapiro [7], among the rich literature on these developments we would like to refer also to [13, 22, 28, 39, 40, 46, 53, 55, 56] and the references therein.

Directional stability of the optimal value function in linear or/and convex semi–infinite programming is studied, e.g., in Zencke and Hettich [58] and Shapiro [54, 55].

Under the so–called reduction ansatz for the constraints, a perturbed semi–infinite program may be regarded (at least locally, i.e., in a neighborhood of a feasible point of interest) as a parametric finite C^2– or $C^{1,1}$–program, for this approach the reader may consult [31]. For example, the results on upper Lipschitz continuity of local minimizers apply, besides the above mentioned papers we also refer to [1, 5, 29, 30, 31, 32, 35]. Similarly, conditions for strong stability of Karush–Kuhn-Tucker points apply, for recent progress which also includes parametric $C^{1,1}$–programs, we refer to Kummer [36]–[38].

Concerning the subjects treated in the present paper, the question arises how to extend the results to the class of generalized semi–infinite programs, i.e., to programs in which the constraint index set may depend also on the state variable x. This question is still under work. Optimality conditions for this class of problems may be found in Hettich and Still [19] and Jongen, Rückmann and Stein [24].vspace0.5cm

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