Non-parametric approximation of non-anticipativity

Idea and test case

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For multistage problems, stochastic programming is usually performed on scenario trees. These trees are used to represent the non-anticipativity constraint and can be difficult to build if only historical scenarios are available.

We try to provide an alternative to scenario trees that makes use of a non-parametric estimate of non-anticipativity.

We present the idea and apply it to an example. Convergence of the scheme is not currently addressed.
Overview of stochastic programming

Approximation of non-anticipativity

Hydro management example
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Hydro management example
Stochastic optimization is difficult problem, even more so in the multistage case [Shapiro, 2006], but appears nevertheless quite frequently in a large number of domains and applications (finance, production, power, logistic, scheduling, . . . ).

Multistage problems differ from two stages ones by their information structure. In the multistage case, not only an optimal decision must be obtained for the first stage, but also decision rules for the following stages, which are functions of the information acquired at each stage.

This progressive acquisition of information and the associated measurability constraints make this problem much harder.
Therefore, contrarily to the two stages case, it is not possible to use a simple Monte Carlo sampling approach.

I’ll put aside dynamic programming based methods which are a separate class with their strengths and weaknesses and I will concentrate on stochastic programming based methods. Among these methods, we can find:

1. Closed loop stochastic gradient methods [Barty et al., 2005] which adapt stochastic gradient techniques to the functional nature of the multistage problems;

2. Methods of progressive refinement of a discrete approximation of the problem [Casey and Sen, 2006];

3. And the most common methods which first discretize the problem completely before solving it using classical means.
Among those methods we can find:

1. Linear decision rules based methods [Thénié and Vial, 2004, Holt et al., 1955] or quantization [Barty, 2004], which discretize the multistage problem into a two stages one;

2. And the most widely used technique usually associated with “Stochastic Programming” which consists in discretizing the problem into a deterministic finite dimension problem, solve by large scale optimization methods, such as linear programming. This discretization is usually performed with the help of a tree, a structure which intrinsically takes into account measurability constraints on the decisions.

While my approach will be slightly different, I’ll first concentrate on that last class of methods.
Two widely different cases can occur:

1. Either we know how to perform conditional sampling on the uncertainties, typically when the processes involved are known, or when the uncertainties are independent. In that case, a large amount of tree building techniques are available, either Monte-Carlo based [Shapiro, 2003, Shapiro, 2006] or Quasi Monte Carlo based [Pennanen and Koivu, 2005].

2. Or we cannot perform such sampling and only scenarios are available, typically when only historical data are available, or the scenario generation procedure is not adapted to conditional sampling (e.g., some physical simulations). In this last case, if the uncertainties are continuous, the scenarios do no lend themselves directly to a tree structure.
In this last case, a few techniques are available (see [Dupačová et al., 2000] for a survey). Mainly:

1. Moment matching techniques
   [Høyland and Wallace, 2001, Høyland et al., 2003];

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Hydro management example
Let $\xi_1$ and $\xi_2$, two random variables of support $\Xi_1 \subset \mathbb{R}^{p_1}$ and $\Xi_2 \subset \mathbb{R}^{p_2}$, and $\Xi_1 \times \Xi_2$ equipped with the euclidean norm denoted $\| \cdot \|$. 

We address the following three stage problem:

$$J = \min_{u_0, u_1(\cdot)} \mathbb{E} [f (u_0, u_1 (\xi_1), \xi_1, \xi_2)]$$  \hspace{1cm} (1)$$

Under the constraints

$$\left\{ \begin{array}{l} u_0 \in U_0 \subset \mathbb{R}^{n_0} \\ \forall \xi_1 \in \Xi_1, \ u_1 (\xi_1) \in U_1 \subset \mathbb{R}^{n_1} \end{array} \right.$$ 

We suppose that for all $u_0 \in U_0$ $u_1 (\xi_1) \in U_1$, $\xi_1$ and $\xi_2$, $f (u_0, u_1 (\xi_1), \xi_1, \xi_2)$ is finite and $f (u_0, u_1 (\xi_1), \xi_1, \xi_2)$ is measurable and integrable for all $u_0$ and $u_1(\cdot)$ considered.
Hypothesis on the dynamic

We suppose that there exists a function $h_1$ such that $\xi_2 = h_1(\xi_1, \varepsilon_2)$ with $\varepsilon_2$ a random variables in $\mathbb{R}^{p'_2}$ independent of $\xi_1$.

Scenarii

Let $\left(\xi^j_1, \xi^j_2, \varepsilon^j_2\right)_{j=1,...,N}$ a sample of $N$ independent drawings of $(\xi_1, \xi_2, \varepsilon_2)$.

We try to solve problem (1), using only the values $\left(\xi^j_1, \xi^j_2\right)_{j=1,...,N}$ that correspond to the historical scenarios observed.

The transfer function $h_1$ is supposed unknown.
Approximation of non-anticipativity
Equivalent formulation

We can express problem (1) equivalently as

$$ J = \min_{u_0, u_1(\cdot)} \mathbb{E} [ f(u_0, u_1, \xi_1, \xi_2) ] $$

Under the constraints

$$\begin{cases}
  u_0 \in U_0 \subset \mathbb{R}^{n_0} \\
  u_1 \in U_1 \subset \mathbb{R}^{n_1}, \text{ a.s.}
\end{cases}$$

and the non-anticipativity constraint on $u_1$:

$$ \mathbb{E} [u_1 | \xi_1] = u_1 $$
Let $D > 0$. We relax the non-anticipativity constraint

$$\mathbb{E}[u_1 | \xi_1] = u_1$$

by

$$\|\mathbb{E}[u_1 | \xi_1] - u_1\|^2 \leq D^2$$

Then replace the conditional expectation by a non-parametric estimate over the scenarios

$$\sum_{j=1,\ldots,N} \left\| \frac{\sum_{k=1,\ldots,N} K\left(\frac{\|\xi_1^j - \xi_1^k\|}{\epsilon}\right) u_1^k}{\sum_{k=1,\ldots,N} K\left(\frac{\|\xi_1^j - \xi_1^k\|}{\epsilon}\right)} - u_1^j \right\|^2 \leq D^2$$

with $K(\cdot)$ a (typically Gaussian) kernel, and $(u_1^j)_{j=1,\ldots,N}$ the realizations of $u_1$ on the historical scenario.
Finally, we approximate the initial problem (1) by the following deterministic problem:

\[
\tilde{J}_N = \min_{u_0, (u^j_1)_{j=1,\ldots,N}} \frac{1}{N} \sum_{j=1,\ldots,N} f \left( u_0, u^j_1, \xi^j_1, \xi^j_2 \right)
\]

subject to the constraints \( u_0 \in U_0 \) and \( \forall j = 1, \ldots, N, \ u^j_1 \in U_1 \) and

\[
\sum_{j=1,\ldots,N} \left\| \frac{\sum_{k=1,\ldots,N} K \left( \frac{\|\xi^j_1 - \xi^k_1\|}{\epsilon} \right) u^j_1}{\sum_{k=1,\ldots,N} K \left( \frac{\|\xi^j_1 - \xi^k_1\|}{\epsilon} \right)} - u^j_1 \right\|^2 \leq D^2
\]

We expect this approximation to converge to the true problem when \( D \to 0, \ \epsilon \to 0 \) and \( N \to \infty \) at the appropriate speeds.
Overview of stochastic programming

Approximation of non-anticipativity

Hydro management example
We consider the problem of managing an hydro-power plant. Two successive production decisions \( u_1 \geq 0 \) and \( u_2 \geq 0 \) have to be made.

The reservoir initially contains an amount of energy \( S \), so that we require \( u_1 + u_2 \leq S \).

These decisions have to be taken as feedbacks on successive random selling prices \( \xi_1 \) and \( \xi_2 \).

There is a non-anticipativity constraint on the first decision, i.e., \( u_1 \) has to be taken prior to any knowledge of the second price, except its conditional law with respect to the first one.
Mathematically, we consider the following cost function:

$$f(u_1, u_2, \xi_1, \xi_2) = -u_1 \xi_1 - u_2 \xi_2 - V(S - u_1 + u_2)$$

where $V(x)$ the value of the remaining stock at the end of the two steps, and is in our case a quadratic approximation of $\sqrt{\eta + x}$, i.e.,

$$V(x) = \sqrt{\eta} + ax + bx^2,$$

with

$$b = \frac{2}{S^2}\left(\sqrt{\eta} - 2\sqrt{\eta + \frac{S}{2}} + \sqrt{\eta + S}\right),$$

$$a = \frac{\sqrt{\eta + S} - \sqrt{\eta} - b}{S}$$

and

$$\eta = 0.1. \quad \xi_1 \text{ and } \xi_2 \text{ follow independent uniform laws on } [0.4, 2],$$

and $S = 1$. 
Our optimization problem is therefore:

\[ J = \min_{u_1(\cdot), u_2(\cdot)} \mathbb{E} \left[ -u_1(\xi_1) \xi_1 - u_2(\xi_1, \xi_2) \xi_2 - V(S - u_1(\xi_1) + u_2(\xi_1, \xi_2)) \right] \]

subject to, \( \forall (\xi_1, \xi_2) \in [0.4, 2]^2 \), \( u_1(\xi_1) \in [0, S] \) and \( u_2(\xi_1, \xi_2) \in [0, S - u_1(\xi_1)] \).

The problem can be solved exactly by dynamic programming. We compare this exact solution to our approximation of the non-anticipativity constraint. Let \( (\xi^j_1, \xi^j_2)_{j=1,\ldots,N} \) be \( N \) independent realizations of the prices.
We consider the following problem:

\[
\tilde{J}_N = \min_{(u_1^j, u_2^j)} \frac{1}{N} \sum_{j=1}^N f \left( u_1^j, u_2^j, \xi_1^j, \xi_2^j \right) + \frac{C}{N} \sum_{j=1}^N \left\| u_1^j - \sum_{k=1}^N K_{\epsilon_1} \left( \xi_1^j, \xi_1^k \right) u_1^k \right\|^2
\]

where \( K_{\epsilon} (x, y) = e^{-\frac{||x-y||^2}{\epsilon^2}} \).
Feedback synthesis

Once this problem is solved, we synthesize the required feedbacks $u_1(\cdot)$ and $u_2(\cdot, \cdot)$ using:

$$u_1(\xi_1) = \frac{\sum_{j=1}^{N} K_{\epsilon_1}(\xi^j_1, \xi_1) u^j_1}{\sum_{j=1}^{N} K_{\epsilon_1}(\xi^j_1, \xi_1)}$$

$$u_2(\xi_1, \xi_2) = \frac{\sum_{j=1}^{N} K_{\epsilon_2}( (\xi^j_1, \xi^j_2), (\xi_1, \xi_2) ) u^j_2}{\sum_{j=1}^{N} K_{\epsilon_2}( (\xi^j_1, \xi^j_2), (\xi_1, \xi_2) )}$$

with $\epsilon_2$ chosen to provide the best fit for the point set $( (\xi^j_1, \xi^j_2), u^j_2 )$. We choose empirically $\epsilon_2 = \sqrt{\frac{\epsilon_1}{\pi}}$. The quality of this approximate is then evaluated by a large Quasi-Monte-Carlo simulation.
Approximation as a function of $C$ and $\epsilon_1$

$\epsilon_1 = 0.02$

$\epsilon_1 = 0.1$

$\epsilon_1 = 0.5$

$C = 1$

$C = 5$

$C = 25$

**Figure:** Approximation of $u_1(\cdot)$ for $N = 100$ scenarios as a function of the bandwidth $\epsilon_1$ and the penalty $C$. 
Approximation as a function of $C$ and $\epsilon_1$

$\epsilon_1 = 0.02$

$\epsilon_1 = 0.1$

$\epsilon_1 = 0.5$

$C = 1$

$C = 5$

$C = 25$

**Figure:** Approximation of $u_2(\cdot, \cdot)$ for $N = 100$ scenarios as a function of the bandwidth $\epsilon_1$ and the penalty $C$. 
Optimal feedback $u_2(\cdot, \cdot)$

**Figure:** Optimal feedback $u_2(\cdot, \cdot)$
The results must be compared to the optimum value, $J = -1.7414$, the value obtained without the penalty term, i.e., the anticipative solution assuming the future is known, whose value is -1.786, and the value of the solution obtained by synthesizing from the anticipative solution, feedbacks as detailed above, with $\epsilon_1 = 0.1$. The value of this last solution is $-1.69563$. 
Approximation as a function of $C$ and $\epsilon_1$

<table>
<thead>
<tr>
<th></th>
<th>$\epsilon_1 = 0.02$</th>
<th>$\epsilon_1 = 0.1$</th>
<th>$\epsilon_1 = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C = 1$</td>
<td>-1.6788</td>
<td>-1.72698</td>
<td>-1.67559</td>
</tr>
<tr>
<td>$C = 5$</td>
<td>-1.67757</td>
<td>-1.73394</td>
<td>-1.66757</td>
</tr>
<tr>
<td>$C = 25$</td>
<td>-1.67638</td>
<td>-1.72863</td>
<td>-1.58707</td>
</tr>
</tbody>
</table>

**Table:** Quality of the approximation for $N = 100$ scenarios as a function of the bandwidth $\epsilon_1$ and the penalty $C$. 
**Case \( C = 0 \)**

**Figure:** Approximations of \( u_1 (\cdot) \) and \( u_2 (\cdot, \cdot) \), for \( N = 100 \) scenarios, with bandwidth \( \epsilon_1 = 0.1 \) and no penalty, i.e., \( C = 0 \).
Best approximations for various $N$

Figure: Best approximation of $u_1(\cdot)$ for $N = 10, 27, 129$ and $999$. 

$N = 10$

$N = 27$

$N = 129$

$N = 999$
Best approximations for various $N$

$N = 10$

$N = 27$

$N = 129$

$N = 999$

**Figure:** Best approximation of $u_2 (\cdot, \cdot)$ for $N = 10, 27, 129$ and $999$. 
Best approximations for various $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\epsilon_1$</th>
<th>$C$</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.215443</td>
<td>5.99484</td>
<td>-1.70561</td>
</tr>
<tr>
<td>27</td>
<td>0.129155</td>
<td>2.15443</td>
<td>-1.72187</td>
</tr>
<tr>
<td>129</td>
<td>0.0774264</td>
<td>5.99484</td>
<td>-1.73369</td>
</tr>
<tr>
<td>999</td>
<td>0.016681</td>
<td>1000</td>
<td>-1.74018</td>
</tr>
</tbody>
</table>

**Table:** Quality and values of $\epsilon_1$ and $C$ for the best solutions for $N = 10$, 27, 129 and 999.
1. It seems to work;
2. It is quite simple to implement, and parameters can be defined through cross-validation or simulation;
3. It should be able to decompose by scenario the problem obtained to speed up the computation;
4. We are working on proving convergence.


