J Optim Theory Appl (2016) 169:925–952
DOI 10.1007/s10957-015-0793-x

# Outer Limit of Subdifferentials and Calmness Moduli in Linear and Nonlinear Programming

**M. J. Cánovas[1] · R. Henrion[2] · M. A. López[3] · J. Parra[1]**

Received: 13 February 2015 / Accepted: 27 July 2015 / Published online: 5 August 2015


**Abstract** With a common background and motivation, the main contributions of this paper are developed in two different directions. Firstly, we are concerned with functions, which are the maximum of a finite amount of continuously differentiable functions of $n$ real variables, paying special attention to the case of polyhedral functions. For these max-functions, we obtain some results about outer limits of subdifferentials, which are applied to derive an upper bound for the calmness modulus of nonlinear systems. When confined to the convex case, in addition, a lower bound on this modulus is also obtained. Secondly, by means of a Karush–Kuhn–Tucker index set approach, we are also able to provide a point-based formula for the calmness modulus of the argmin mapping of linear programming problems, without any uniqueness assumption on the optimal set. This formula still provides a lower bound in linear semi-infinite programming. Illustrative examples are given.

✉ M. A. López
marco.antonio@ua.es

M. J. Cánovas
canovas@umh.es

R. Henrion
henrion@wias-berlin.de

J. Parra
parra@umh.es

1 Center of Operations Research, Miguel Hernández University of Elche, Elche (Alicante), Spain

2 Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany

3 Department of Statistics and Operations Research, University of Alicante, 03071 Alicante, Spain

**Keywords** Calmness · Local error bounds · Variational analysis · Linear programming · Argmin mapping

**Mathematics Subject Classification** 90C31 · 49J53 · 94K40 · 90C25 · 90C05

## 1 Introduction

The present paper, was initially motivated by the problem of computing the *calmness modulus* of linear programs having optimal sets which are not a singleton. In relation to this problem, the immediate antecedents are gathered in [1–3], where the assumption of the uniqueness of nominal optimal solution is essential. To this respect, we advance that an exact formula for the aimed modulus is obtained in Sect. 4 and that it is given in terms of the calmness moduli of certain sublevel multifunctions, which are nothing else but feasible set mappings.

In the context of finite linear systems, the computation of the calmness modulus for feasible set mappings is dealt in [4], where an operative expression (exclusively in terms of the nominal data) for this modulus is provided. With respect to this subject, the present work presents some extensions to the setting of $\mathcal{C}^1$-systems, where the constraints are described by continuously differentiable (sometimes convex) functions.

According to Theorem 2.1 below, the key ingredient in the computation (or estimation) of the calmness modulus for a $\mathcal{C}^1$-system at some feasible point is the outer limit of subdifferentials, by approaching this point from outside the feasible set, of a certain max-function associated with the system. Besides this original motivation and its application to calmness moduli, the problem of analyzing this outer limit is of independent interest, and it is tackled in the present paper in two stages: firstly, in the particular case of polyhedral functions and, in a second step, in the more general context of continuously differentiable functions. The reader is addressed to [1, Theorem 3.1] for a direct antecedent to this problem, when confined to the convex case (not necessarily differentiable).

The results about outer limits of subdifferentials obtained in the current work are applied to derive an upper bound on the calmness modulus of the feasible set mapping associated with a parameterized $\mathcal{C}^1$-system, under *right-hand-side* (RHS) *perturbations*. If, additionally, functions defining the constraints are convex, then we also derive a lower bound on the aimed calmness modulus. These results are inspired by the known exact formula for linear systems, which is recalled in Theorem 2.2 for completeness purposes. In this case of finite linear systems, it is well known that the feasible set mapping is always calm at any point of its (polyhedral) graph as a consequence of a classical result by Robinson [5].

The paper also deals with the calmness of the *optimal set mapping* (also called *argmin mapping*), $\mathcal{S}$, in the framework of linear problems with *canonical perturbations*, i.e., where perturbations fall on the objective function coefficient vector and on the RHS of the constraints. The same result by Robinson ensures that mapping $\mathcal{S}$ is always calm at any point of its graph, since the Karush–Kuhn–Tucker (KKT) conditions allow us to express the graph of $\mathcal{S}$ as a finite union of polyhedral sets. This is no longer the case in the framework of perturbations of all data. In relation to this last framework, [2, Theorem 4.1] establishes a characterization for the calmness of the corresponding argmin mapping (by combining two results from the seminal paper [6]) and provides an operative upper bound for the corresponding calmness modulus, assuming the uniqueness of nominal optimal solution.

Comprehensive studies on calmness and other variational properties for generic multifunctions can be traced out from the monographs [7–10]; see also [11–14] in relation to the calmness of constraint systems in the context of RHS perturbations; where calmness translates into the existence of a *local error bound* for the corresponding supremum function (see [15–17]). Other subdifferential approaches to calmness/local error bounds can be found in [18,19].

The structure of the paper is as follows: Sect. 2 provides the necessary notation, definitions and preliminary results. Section 3 gathers the announced results on outer limits of subdifferentials of max-functions under different assumptions. It is divided into three subsections. The first one deals with the particular case of a polyhedral function, where an exact formula is provided, while the second is focused on the nonlinear case. The third subsection provides the application to the estimation of the calmness modulus for the feasible set mapping in the context of $\mathcal{C}^1$-systems mentioned above, paying attention to the particular case of convex $\mathcal{C}^1$-systems. In Sect. 4, by means of a KKT index set approach, we provide an operative expression for the calmness modulus of $\mathcal{S}$ at a given nominal parameter in the case when the nominal optimal set does not necessarily confine itself to a singleton. Moreover, we prove that this expression still remains as a lower bound in the semi-infinite continuous case, when the index set $T$ is assumed to be a compact Hausdorff space and all the constraints' coefficients are continuous functions (with respect to the index) on $T$. The reader is addressed to [20, Chapter 10] and [21] for details about stability in this semi-infinite setting. Illustrative examples are provided in order to show that, in this general case (without uniqueness of nominal optimal solution), the referred expression may be strictly smaller than the upper bound given in [3, Theorem 7]. Section 5 offers some perspectives for future research and specifies some open problems to this respect. We finish the paper with a section of conclusions (Sect. 6).

## 2 Preliminaries

In this section, we introduce some notation, definitions and preliminary results which are needed later on. Given $A \subseteq \mathbb{R}^k$, we denote by $\operatorname{conv} A$ and $\operatorname{cone} A$ the *convex hull* and the *conical convex hull* of $A$, respectively. It is assumed that $\operatorname{cone} A$ always contains the zero vector $0_k$, in particular $\operatorname{cone}(\emptyset) = \{0_k\}$. If $A$ is a subset of any topological space, $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$ stand, respectively, for the (topological) interior, the closure and the boundary of $A$. If $\|\cdot\|$ is any norm in $\mathbb{R}^k$, its corresponding *dual norm* is denoted by $\|\cdot\|_*$, i.e., $\|u\|_* = \max_{\|x\| \leq 1} \left|u'x\right|$.

In the next paragraphs, we recall some definitions related to a generic mapping $\mathcal{M} : Y \rightrightarrows X$ between metric spaces (with distances denoted indistinctly by $d$). $\mathcal{M}$ is said to be *calm* at $(\overline{y}, \overline{x}) \in \operatorname{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) iff there exist a constant $\kappa \geq 0$ and neighborhoods $U$ of $\overline{x}$ and $V$ of $\overline{y}$ such that

$$d\left(x,\mathcal{M}(\overline{y})\right)\leq\kappa d\left(y,\overline{y}\right) \tag{1}$$

whenever $x\in\mathcal{M}(y)\cap U$ and $y\in V$; where, as usual, $d(x,\Omega)$ is defined as $\inf\{d(x,z):z\in\Omega\}$ for $\Omega\subseteq\mathbb{R}^n$, and $d(x,\emptyset):=+\infty$.

It is well known that the calmness of $\mathcal{M}$ at $(\overline{y},\overline{x})$ is equivalent to the *metric subregularity* of the inverse multifunction $\mathcal{M}^{-1}$ at $(\overline{x},\overline{y})$ (see, for instance, [7, Theorem 3H.3 and Exercise 3H.4]), which reads as follows: there exist a constant $\kappa\geq 0$ and a (possibly smaller) neighborhood $U$ of $\overline{x}$ such that

$$d\left(x,\mathcal{M}(\overline{y})\right)\leq\kappa d\left(\overline{y},\mathcal{M}^{-1}(x)\right),\quad\text{for all } x\in U. \tag{2}$$

The infimum of those $\kappa\geq 0$ for which (1)—or (2)—holds (for some associated neighborhoods) is called the *calmness modulus* of $\mathcal{M}$ at $(\overline{y},\overline{x})$ and is denoted by $\operatorname{clm}\mathcal{M}(\overline{y},\overline{x})$. The case $\operatorname{clm}\mathcal{M}(\overline{y},\overline{x})=+\infty$ corresponds to the one in which $\mathcal{M}$ is not calm at $(\overline{y},\overline{x})$.

### 2.1 Preliminaries on the Feasible Set Mapping

We consider the parametrized $\mathcal{C}^1$-system

$$\sigma(b):=\{f_i(x)\leq b_i,\quad\text{for all } i=1,\ldots,m\}, \tag{3}$$

and the associated feasible set mapping $\mathcal{F}:\mathbb{R}^m\rightrightarrows\mathbb{R}^n$, given by

$$\mathcal{F}(b):=\left\{x\in\mathbb{R}^n:\ f_i(x)\leq b_i,\quad\text{for all } i=1,\ldots,m\right\}, \tag{4}$$

where $f_i\in\mathcal{C}^1(\mathbb{R}^n)$ and $b_i\in\mathbb{R}$, $i=1,\ldots,m$. In this setting, $b\equiv(b_i)_{i=1,\ldots,m}$ is the parameter to be perturbed. The space of variables of the system, $\mathbb{R}^n$, is equipped with an arbitrary norm, while our *parameter space*, $\mathbb{R}^m$, is endowed with the supremum norm $\|b\|_\infty:=\max_{i=1,\ldots,m}|b_i|$, $b\in\mathbb{R}^m$.

Associated with system (3), for a given nominal parameter $\overline{b}$, we consider the max-function

$$g:=\max_{1,\ldots,m} g_i,\quad\text{where } g_i(x)=f_i(x)-\overline{b}_i,\quad i=1,\ldots,m. \tag{5}$$

Throughout the paper, we appeal to the *set of active indices* at $x\in\mathcal{F}(b)$, denoted by $T_b(x)$ and defined as

$$T_b(x):=\{i\in\{1,\ldots,m\}:\ f_i(x)=b_i\}.$$

If $T_b(x)=\emptyset$, $x$ is a Slater point of $\sigma(b)$, and in this case, one trivially has $\operatorname{clm}\mathcal{F}(b,x)=0$. So, along the paper we assume that our nominal solution $\overline{x}\in\mathcal{F}\left(\overline{b}\right)$ satisfies $T_{\overline{b}}(\overline{x})\neq\emptyset$, or, equivalently,

$$g(\overline{x})=0.$$

The following theorem constitutes our starting point in the estimation of $\text{clm}\mathcal{F}(\overline{b}, \overline{x})$. Statement (i) in this theorem comes from [16, Propositions 1, 11 and 5(ii)], whereas (ii) follows directly from [17, Theorem 1]. In it, we have taken into account the well-known relationship between $\text{clm}\mathcal{F}(\overline{b}, \overline{x})$ and the *error bound modulus* of $g$ at $\overline{x}$, specifically

$$\text{clm}\mathcal{F}(\overline{b}, \overline{x}) = [\text{Er}\, g(\overline{x})]^{-1}, \tag{6}$$

and the easily verifiable fact that

$$\liminf_{x \to \overline{x},\ g(x)>0} d_* (0_n, \partial g(x)) = d_* \left( 0_n, \limsup_{x \to \overline{x},\ g(x)>0} \partial g(x) \right), \tag{7}$$

where $d_*$ stands for the distance in $\mathbb{R}^n$ associated with $\|\cdot\|_*$ and $\partial g$ represents the Clarke subdifferential of $g$.

**Theorem 2.1** *Let* $\left(\overline{b}, \overline{x}\right) \in \text{gph}\mathcal{F}$ *such that* $g(\overline{x}) = 0$. *Then*:

(i) *We have*

$$\text{clm}\mathcal{F}(\overline{b}, \overline{x}) \leq \left[ d_* \left( 0_n, \limsup_{x \to \overline{x},\ g(x)>0} \partial g(x) \right) \right]^{-1}; \tag{8}$$

(ii) *If, additionally, functions* $f_i$ *in* (5), $i = 1, \ldots, m$, *are convex, then*

$$\text{clm}\mathcal{F}(\overline{b}, \overline{x}) = \left[ d_* \left( 0_n, \limsup_{x \to \overline{x},\ g(x)>0} \partial g(x) \right) \right]^{-1}. \tag{9}$$

*Remark 2.1* In relation to the previous theorem, let us comment that:

(i) With respect statement (i), [16, Propositions 1, 11 and 5(ii)] refers to the Fréchet subdifferential $\hat{\partial}$. In principle, from that results in [16] we deduce

$$\text{Er}\, g(\bar{x}) \geq \liminf_{x \to \bar{x}, g(x)>0} d_* \left( 0_n, \hat{\partial} g(x) \right).$$

However, in our case we may replace $\hat{\partial}$ by the Clarke subdifferential, $\partial$, as consequence of the Clarke regularity of $g$ (see the beginning of Sect. 3), and then, taking also (6) and (7) into account, we obtain inequality (8).

(ii) Equality (9) is held under convexity, even without differentiability assumptions on the $f_i$'s (see again [17, Theorem 1]), in which case, $\partial g$ stands for the usual subdifferential of convex analysis.

Our next step is to obtain estimations for $\text{clm}\mathcal{F}(\overline{b}, \overline{x})$ which only involve the nominal data. Having the previous theorem in mind, as advanced in Sect. 1, a way of tackling this problem consists of analyzing the outer limit inside. The next theorem, dealing with the case of linear systems, provides a motivation for some results of the following section (Theorems 3.1 and 3.2).

The following theorem deals with the linear case, in which the $f_i$'s are given by

$$f_i(x) := a_i' x,\ i = 1, \ldots, m,$$

where $a_i \in \mathbb{R}^n$, $1, \ldots, m$, are fixed. Here, any vector $y \in \mathbb{R}^n$ is regarded as a column vector, and $y'$ denotes its transpose (hence, $y'x$ stands for the usual inner product). In order to emphasize the difference between the linear and nonlinear contexts, the feasible set mapping in the particular case of linear systems will be denoted by $\mathcal{F}_a$; specifically,

$$\mathcal{F}_a(b) := \left\{x \in \mathbb{R}^n : a_i' x \leq b_i, \quad \text{for all } i = 1, \ldots, m\right\}. \tag{10}$$

From now on, $\mathcal{D}_{\overline{b}}(\overline{x})$ denotes the family of all subsets $D \subseteq T_{\overline{b}}(\overline{x})$ such that system

$$\left\{ \begin{array}{l} a_i' d = 1,\ i \in D, \\ a_i' d < 1,\ i \in T_{\overline{b}}(\overline{x}) \setminus D \end{array} \right\} \tag{11}$$

is consistent (in the variable $d \in \mathbb{R}^n$). In other words, $D \in \mathcal{D}_{\overline{b}}(\overline{x})$ iff there exists a hyperplane containing $\{a_i, i \in D\}$ such that

$$\{0_n\} \cup \{a_i, i \in T_{\overline{b}}(\overline{x}) \setminus D\}$$

lies on one of the open half-spaces determined by this hyperplane.

**Theorem 2.2** [4, Theorem 4] *Given* $\left(\overline{b}, \overline{x}\right) \in \mathrm{gph}\mathcal{F}$, *we have*

$$\mathrm{clm}\mathcal{F}_a(\overline{b}, \overline{x}) = \left(\min_{D \in \mathcal{D}_{\overline{b}}(\overline{x})} d_* \left(0_n, \mathrm{conv}\left\{a_i,\ i \in D\right\}\right)\right)^{-1}.$$

### 2.2 Preliminaries on the Argmin Mapping

We consider the optimal set mapping $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ in the linear framework under canonical perturbations, which is given by

$$\mathcal{S}(c, b) := \arg\min\{c'x : x \in \mathcal{F}_a(b)\}. \tag{12}$$

The parameter space, $\mathbb{R}^n \times \mathbb{R}^m$, is endowed with the norm

$$\|(c, b)\| := \max\left\{\|c\|_*, \|b\|_\infty\right\},\ (c, b) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{13}$$

The next theorem comes directly from [3, Theorem 7] and constitutes our starting point of Sect. 4. In it, associated with a given $\left((\overline{c}, \overline{b}), \overline{x}\right) \in \mathrm{gph}\mathcal{S}$, we appeal to the following family of index subsets associated with the KKT conditions (hereafter referred to as *KKT index sets*):

$$\mathcal{K}_{\overline{c},\overline{b}}(\overline{x}) = \left\{D \subseteq T_{\overline{b}}(\overline{x}) : |D| \leq n \text{ and } -\overline{c} \in \mathrm{cone}\left\{a_i,\ i \in D\right\}\right\},$$

where $|D|$ stands for the cardinality of $D$ and condition $|D| \leq n$ comes from Carathéodory's Theorem. For any $D \in \mathcal{K}_{\overline{c},\overline{b}}(\overline{x})$, we consider the mapping $\mathcal{L}_D : \mathbb{R}^m \times \mathbb{R}^D \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{L}_D(b,d) := \left\{x \in \mathbb{R}^n : a_i' x \leq b_i,\ i = 1,\ldots,m;\ -a_i' x \leq d_i,\ i \in D\right\}. \tag{14}$$

Observe that all preliminary results for the feasible set mappings $\mathcal{F}_a$ may be specified for $\mathcal{L}_D$, which is nothing else but the feasible set mapping associated with an enlarged system.

**Theorem 2.3** [3, Theorem 7] *Let* $(\overline{c},\overline{b}) \in \mathbb{R}^n \times \mathbb{R}^m$. *Then*

$$\operatorname{clm}\mathcal{S}((\overline{c},\overline{b}),\overline{x}) = \operatorname{clm}\mathcal{S}_{\overline{c}}(\overline{b},\overline{x}) \leq \max_{D\in\mathcal{K}_{\overline{c},\overline{b}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left((\overline{b},-\overline{b}_D),\overline{x}\right), \tag{15}$$

*where* $\overline{b}_D$ *stands for* $(\overline{b}_i)_{i\in D}$ *and* $\mathcal{S}_{\overline{c}}(b) := \mathcal{S}(\overline{c},b)$ for $b \in \mathbb{R}^m$.

*Remark 2.2* Corollary 8 in [3] shows that (15) holds as an equality under the additional assumption that $\mathcal{S}(\overline{c},\overline{b}) = \{\overline{x}\}$.

## 3 Outer Limits of Subdifferentials and Calmness Modulus of Differentiable Nonlinear Systems

The present section is divided into three subsections. The first one, inspired by Theorem 2.2 (having also Theorem 2.1 in mind), establishes an exact expression for the outer limit of subdifferentials of polyhedral functions. The second is focused on the more general case of max-functions in a nonlinear differentiable framework. The third applies some previous results to obtain estimations of the aimed $\operatorname{clm}\mathcal{F}(\overline{b},\overline{x})$ for convex and nonlinear systems.

We consider the max-function $g : \mathbb{R}^n \to \mathbb{R}$

$$g(x) := \max_{i=1,\ldots,m} g_i(x),$$

where the $g_i$'s are continuously differentiable on $\mathbb{R}^n$. As a consequence, $g$ is a regular function in the sense of Clarke (see, for instance, [10, Examples 10.24(e) and 10.25(a)]), and we have

$$\partial g(x) = \operatorname{conv}\{\nabla g_i(x) : i \in I(x)\}$$

(see [22, 2.3.12]), where

$$I(x) := \{i = 1,\ldots,m : g_i(x) = g(x)\}.$$

Note that always $I(x) \neq \emptyset$. It is a known fact that for each $x \in \mathbb{R}^n$, there exists $\varepsilon_x > 0$ such that

$$I(z) \subseteq I(x) \quad \text{whenever } \|z - x\| < \varepsilon_x. \tag{16}$$

Finally, inspired by $\mathcal{D}_{\overline{b}}(\overline{x})$ (see (11)) we define the family $\mathcal{D}(\overline{x})$ formed by all subsets of indices $D \subseteq I(\overline{x})$ such that the system

$$\left\{ \begin{array}{ll} \nabla g_i(\overline{x})' d = 1, & i \in D, \\ \nabla g_i(\overline{x})' d < 1, & i \in I(\overline{x}) \setminus D \end{array} \right\} \tag{17}$$

is consistent in the variable $d \in \mathbb{R}^n$.

### 3.1 Outer Limits of Subdifferentials of Polyhedral Functions

This subsection deals with the particular case when the $g_i$'s are affine functions, i.e.,

$$g_i(x) := a_i' x - b_i, \quad i = 1, \ldots, m,$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ are fixed. In this case, the corresponding max-function

$$g(x) := \max_{i=1,\ldots,m} \{a_i' x - b_i\}, \tag{18}$$

is a *polyhedral function*, and its subdifferential (in the sense of Clarke, which in this case coincides with the usual subdifferential of convex analysis) writes as

$$\partial g(x) = \operatorname{conv}\{a_i : i \in I(x)\}. \tag{19}$$

**Theorem 3.1** *Let $g$ be defined in (18). We have*

$$\limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x) = \bigcup_{D \in \mathcal{D}(\overline{x})} \operatorname{conv}\{a_i,\ i \in D\}.$$

*Proof* First, let us prove the '$\supseteq$' inclusion. Pick any $D \in \mathcal{D}(\overline{x})$ and consider $d \in \mathbb{R}^n$ such that (17) fulfills. Then, for any $\alpha > 0$ one has

$$\left. \begin{array}{ll} g_i(\overline{x} + \alpha d) = a_i'(\overline{x} + \alpha d) - b_i = g_i(\overline{x}) + \alpha = g(\overline{x}) + \alpha & \text{for } i \in D \\ g_i(\overline{x} + \alpha d) = a_i'(\overline{x} + \alpha d) - b_i < g_i(\overline{x}) + \alpha = g(\overline{x}) + \alpha & \text{for } i \in I(\overline{x}) \setminus D \end{array} \right\}. \tag{20}$$

Suppose $0 < \alpha \|d\| < \varepsilon_{\overline{x}}$, with $\varepsilon_{\overline{x}}$ being as in (16). Then, (20) ensures $g(\overline{x} + \alpha d) = g(\overline{x}) + \alpha$ and

$$\partial g(\overline{x} + \alpha d) = \operatorname{conv}\{a_i,\ i \in D\}.$$

Therefore

$$\operatorname{conv}\{a_i,\ i \in D\} = \limsup_{\alpha \to 0^+} \partial g(\overline{x} + \alpha d) \subseteq \limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x).$$

In order to prove the '$\subseteq$' inclusion, take any $u \in \limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x)$. Let us write $u = \lim_{k\to\infty} u_k$ with $u_k \in \partial g(x_k), g(x_k) > g(\overline{x})$ (for all $k \in \mathbb{N}$) and $x_k \to \overline{x}$ (without loss of generality we assume $\|x_k - \overline{x}\| < \varepsilon_{\overline{x}}$ for all $k$). Then, the sequence $(I(x_k))_{k\in\mathbb{N}}$ has a constant subsequence because $I(x_k) \subseteq \{1, \ldots, m\}$ for all $k$. Accordingly, let us assume without loss of generality that

$$I(x_k) = D \subseteq I(\overline{x}) \quad \text{for all } k \in \mathbb{N}.$$

Since $\partial g(x_k) = \operatorname{conv}\{a_i,\ i \in D\}$ is a compact set (independent on $k$), we obtain

$$u \in \operatorname{conv}\{a_i,\ i \in D\}. \tag{21}$$

Pick any particular $k \in \mathbb{N}$ and define

$$d := \frac{x_k - \overline{x}}{g(x_k) - g(\overline{x})}.$$

Then, we have

$$a_i' d = \frac{a_i' x_k - a_i' \overline{x}}{g(x_k) - g(\overline{x})} = \frac{g_i(x_k) - g_i(\overline{x})}{g(x_k) - g(\overline{x})} \begin{cases} = 1 & \text{for all } i \in D, \\ < 1 & \text{for all } i \in I(\overline{x}) \setminus D. \end{cases}$$

Accordingly, $D \in \mathcal{D}(\overline{x})$, and the proof ends by appealing to (21). □

*Remark 3.1* Since (17) clearly implies that, for each $D \in \mathcal{D}(\overline{x})$, $\operatorname{conv}\{a_i, i \in D\}$ is contained in a supporting hyperplane to $\operatorname{conv}\{a_i, i \in I(\overline{x})\}$, it follows that

$$\begin{aligned} \limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x) &= \bigcup_{D \in \mathcal{D}(\overline{x})} \operatorname{conv}\{a_i,\ i \in D\} \\ &\subseteq \operatorname{bd} \operatorname{conv}\{a_i,\ i \in I(\overline{x})\}) = \operatorname{bd} \partial g(\overline{x}). \end{aligned}$$

The next example shows that the previous inclusion may be strict.

*Example 3.1* (see [4, Example 4]) Consider the system (in $\mathbb{R}^2$ endowed with the Euclidean norm)

$$\{x_1 \leq b_1,\ x_2 \leq b_2,\ x_1 + x_2 \leq b_3\},$$

and the nominal data $\overline{b} = 0_3$ and $\overline{x} = 0_2$. The associated supremum function is given by

$$g(x) = \max\{x_1,\ x_2,\ x_1 + x_2\},$$

and accordingly

$$\operatorname{bd} \operatorname{conv}\{a_i,\ i \in I(\overline{x})\} = \operatorname{conv}\{a_1,\ a_2\} \cup \operatorname{conv}\{a_1,\ a_3\} \cup \operatorname{conv}\{a_2,\ a_3\}.$$

However,

$$\bigcup_{D\in\mathcal{D}(\overline{x})} \operatorname{conv}\{a_i,\ i\in D\} = \operatorname{conv}\{a_1,\ a_3\}\cup\operatorname{conv}\{a_2,\ a_3\}.$$

### 3.2 Extensions to the Nonlinear Differentiable Case

In the following theorem, associated with a fixed point $\overline{x}\in\mathbb{R}^n$, we appeal to the new family of subsets of indices

$$\mathcal{D}_{AI}(\overline{x})\subseteq\mathcal{D}(\overline{x}), \tag{22}$$

formed by all $D\in\mathcal{D}(\overline{x})$ such that $\{\nabla g_i(\overline{x}),\ i\in D\}$ is affinely independent. We also appeal to the family $\mathcal{D}^0(\overline{x})$ defined by replacing 1 with 0 in the definition of $\mathcal{D}(\overline{x})$, see (17), with $d$ being nonzero.

**Theorem 3.2** *Let $g(x):=\max\limits_{i=1,\ldots,m} g_i(x)$, with $g_i:\mathbb{R}^n\to\mathbb{R}$ continuously differentiable for all $i$, and let $\overline{x}\in\mathbb{R}^n$. We have:*

(i) $\bigcup_{D\in\mathcal{D}_{AI}(\overline{x})}\operatorname{conv}\{\nabla g_i(\overline{x}),\ i\in D\}\subseteq\limsup\limits_{x\to\overline{x},\ g(x)>g(\overline{x})}\partial g(x)$
$\subseteq\bigcup_{D\in\mathcal{D}(\overline{x})\cup\mathcal{D}^0(\overline{x})}\operatorname{conv}\{\nabla g_i(\overline{x}),\ i\in D\}\subseteq\operatorname{bd}\partial g(\overline{x})$;

(ii) $\limsup\limits_{x\to\overline{x},\ x\neq\overline{x}}\partial g(x)\subseteq\operatorname{bd}\partial g(\overline{x})$.

*Moreover, the converse inclusion of (ii) also holds if, for all supporting hyperplane $H$ to $\partial g(\bar{x})$, we have that $\{\nabla g_i(\bar{x}), i\in I(\bar{x})\}\cap H$ is affinely independent.*

*Proof* (i) Take any $D\in\mathcal{D}_{AI}(\overline{x})$. For the sake of simplicity, let us assume that $D:=\{1,2,\ldots,i_0\}$ ($i_0\leq n+1$ because of the definition of $\mathcal{D}_{AI}(\overline{x})$) and consider the system of equations

$$\{h_i(x):=g_{i+1}(x)-g_1(x)=0,\quad i=1,\ldots,i_0-1\}. \tag{23}$$

Since $D\in\mathcal{D}_{AI}(\overline{x})$, there exists $d\in\mathbb{R}^n$ such that

$$\nabla g_i(\overline{x})'d=1,\quad i=1,\ldots,i_0, \tag{24}$$

which entails

$$\nabla h_i(\overline{x})'d=0,\quad i=1,\ldots,i_0-1.$$

Moreover, vectors $\nabla h_i(\overline{x})$, $i=1,\ldots,i_0-1$, are linearly independent and $d\neq 0_n$ because of (24), which actually entails $i_0\leq n$. If we write system (23) in the vectorial form

$$h(x)=0_{i_0-1}, \tag{25}$$

observe that $\overline{x}$ is a regular point of the surface $S$ defined by (25). Moreover, if we denote by $\nabla h\,(\overline{x})$ the matrix whose columns are $\nabla h_1\,(\overline{x})\,, \ldots, \nabla h_{i_0-1}\,(\overline{x})$, we have

$$\nabla h\,(\overline{x})'\,d = 0_{i_0-1}.$$

Then, there exists a differentiable curve $\alpha$ such that the arc

$$\{\alpha\,(t)\,,\ -a < t < a\} \subseteq S,\ (a > 0) \tag{26}$$

verifies

$$\alpha(0) = \overline{x} \text{ and } \dot{\alpha}(0) = d$$

(see, for example, [23, p. 325]), where we use a dot standing for derivatives (recall that we are using the prime for transposition). Let us consider a sequence of scalars $0 < t_k < a$, $k \in \mathbb{N}$ such that $t_k \to 0$ and define

$$x_k := \alpha\,(t_k)\,, \quad \text{for all } k.$$

From (26), we have

$$g_1\,(x_k) = \cdots = g_{i_0}\,(x_k)\,, \quad k \in \mathbb{N}. \tag{27}$$

Let $j \in I\,(\overline{x}) \diagdown D$ and consider the function

$$h_j := g_1 - g_j.$$

Observe that

$$\nabla h_j\,(\alpha(0))'\,\dot{\alpha}(0) = \nabla h_j\,(\overline{x})'\,d > 0,$$

which entails (by a standard argument of differential calculus)

$$h_j\,(x_k) = g_1\,(x_k) - g_j\,(x_k) > 0$$

for $k$ sufficiently large. By the previous inequality and taking (27) into account, recall also (16), we have that

$$I\,(x_k) = D, \quad \text{for } k \text{ large enough}, \tag{28}$$

and then

$$\partial g\,(x_k) = \operatorname{conv}\{\nabla g_i\,(x_k)\,, \quad i \in D\}\,.$$

Moreover, for $i \in D$, we have that

$$g_i\,(x_k) > g_i\,(\overline{x})\,, \quad \text{for } k \text{ large enough},$$

since $\nabla g_i(\overline{x})' d = 1 > 0$. Then,

$$g(x_k) > g(\overline{x}), \quad \text{for } k \text{ large enough}$$

[recall (28)]. Finally,

$$\lim_{k\to\infty} \partial g(x_k) = \lim_{k\to\infty} \operatorname{conv}\{\nabla g_i(x_k),\ i \in D\} = \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\}$$

(with the limits being understood in the Painlevé–Kuratowski sense), yielding the aimed inclusion

$$\operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\} \subseteq \limsup_{x\to\overline{x},\ g(x)>g(\overline{x})} \partial g(x).$$

Now let us prove the second inclusion of (i). Take $u \in \limsup_{x\to\overline{x},\ g(x)>g(\overline{x})} \partial g(x)$ written in the form $u = \lim_{k\to\infty} u_k$ with $u_k \in \partial g(x_k)$, $g(x_k) > 0$, and $x_k \to \overline{x}$ as $k \to \infty$. Recall that, for each $k \in \mathbb{N}$, $\partial g(x_k) = \operatorname{conv}\{\nabla g_i(x_k), i \in I(x_k)\}$, and we can assume without loss of generality that $I(x_k) = D$ (independent of $k$) for all $k$, with $D \subseteq I(\overline{x})$. Then, we prove that $u \in \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\}$. In fact, from

$$u_k \in \partial g(x_k) = \operatorname{conv}\{\nabla g_i(x_k) \mid i \in D\}$$

it follows that $u_k = \sum_{i\in D} \lambda_i^k \nabla g_i(x_k)$ for some sequence $\lambda^k \in \mathbb{R}_+^D$ satisfying $\sum_{i\in D} \lambda_i^k = 1$ for all $k$. By compactness of the standard simplex in $\mathbb{R}^D$, we may assume again without loss of generality that, upon passing to another subsequence which we do not relabel, there exists some $\bar{\lambda} \in \mathbb{R}_+^D$ satisfying $\sum_{i\in D} \bar{\lambda}_i = 1$ such that $\lambda^k \to \bar{\lambda}$. Consequently,

$$u_k \to \sum_{i\in D} \bar{\lambda}_i \nabla g_i(\bar{x}) = u,$$

showing that

$$u \in \operatorname{conv}\{\nabla g_i(\bar{x}), \quad i \in D\}. \tag{29}$$

Let us see now that $D$ is contained in a certain member of the family $\mathcal{D}(\overline{x}) \cup \mathcal{D}^0(\overline{x})$. It is not restrictive to assume that (for a suitable subsequence, without relabeling) $\frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to z$ for a certain $z \in \mathbb{R}^n$ with $\|z\| = 1$. Then, for any $i \in D$ and any $k \in \mathbb{N}$ we have

$$\begin{aligned} 0 &< \frac{g(x_k) - g(\overline{x})}{\|x_k - \bar{x}\|} = \frac{g_i(x_k) - g_i(\bar{x})}{\|x_k - \bar{x}\|} \\ &= \nabla g_i(\bar{x})' \left( \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \right) + \frac{o(\|x_k - \bar{x}\|)}{\|x_k - \bar{x}\|} \to \nabla g_i(\bar{x})' z =: \alpha, \end{aligned} \tag{30}$$

where

$$\frac{o(\|x_k - \bar{x}\|)}{\|x_k - \bar{x}\|} \to 0.$$

Observe that $\alpha$ does not depend on $i \in D$. On the other hand, for any $j \in I(\bar{x}) \backslash D$ and any $k \in \mathbb{N}$, we have

$$\nabla g_j(\bar{x})' \left( \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \right) + \frac{o(\|x_k - \bar{x}\|)}{\|x_k - \bar{x}\|} = \frac{g_j(x_k) - g_j(\bar{x})}{\|x_k - \bar{x}\|} \leq \frac{g(x_k) - g(\bar{x})}{\|x_k - \bar{x}\|},$$

and letting $k \to \infty$ in both sides we have $\nabla g_j(\bar{x})'z \leq \alpha$. Let us consider first the case $\alpha > 0$. Then, defining

$$d := z/\alpha = \lim_{k \to \infty} (x_k - \bar{x}) / (g(x_k) - g(\bar{x})),$$

we immediately check that

$$D \subseteq \left\{ i \in I(\bar{x}) : \nabla g_j(\bar{x})'d = 1 \right\} \in \mathcal{D}(\bar{x}).$$

In the case $\alpha = 0$, we directly have $D \subseteq \left\{ i \in I(\bar{x}) : \nabla g_j(\bar{x})'z = 0 \right\} \in \mathcal{D}^0(\bar{x})$.

The last inclusion of (i) comes straightforwardly from the fact that each conv $\{\nabla g_i(\bar{x}), i \in D\}$ with $D \in \mathcal{D}(\bar{x}) \cup \mathcal{D}^0(\bar{x})$ is contained in a supporting hyperplane to $\partial g(\bar{x})$.

(ii) Let $u \in \limsup_{x \to \bar{x}, x \neq \bar{x}} \partial g(x)$ be arbitrary. By definition, there exists a sequence $(x_k, u_k) \to (\bar{x}, u)$ such that $x_k \neq \bar{x}$ and $u_k \in \partial g(x_k)$. After passing to a subsequence which we do not relabel, we may assume without loss of generality that there exists an index set $I \subseteq I(\bar{x})$ and a vector $d \neq 0_n$ such that

$$I(x_k) = I \;\; \forall k \quad \text{and} \quad \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to d.$$

Here, the first statement follows from the fact that $I(x_k) \subseteq I(\bar{x})$ for $k$ large enough and that there exist only finitely many subsets of $I(\bar{x})$. From

$$u_k \in \partial g(x_k) = \text{conv}\{\nabla g_i(x_k) \mid i \in I(x_k)\} = \text{conv}\{\nabla g_i(x_k) \mid i \in I\}$$

it follows again that

$$u \in \text{conv}\{\nabla g_i(\bar{x}), i \in I\}. \tag{31}$$

Next, we prove the following relation involving the vector $d$ introduced above:

$$\nabla g_i(\bar{x})'d \geq \nabla g_j(\bar{x})'d \quad \forall i \in I \; \forall j \in I(\bar{x}). \tag{32}$$

Indeed, if, reasoning by contradiction, (32) fails, then there exist $i \in I$ and $j \in I(\bar{x})$ such that $\nabla g_i(\bar{x})'d < \nabla g_j(\bar{x})'d$. Observe that $I \subseteq I(\bar{x})$ implies that $g_i(\bar{x}) = g_j(\bar{x})$. As $x_k \neq \bar{x}$, we get that

$$\frac{g_i(x_k) - g_j(x_k)}{\|x_k - \bar{x}\|} = \frac{g_i(x_k) - g_i(\bar{x})}{\|x_k - \bar{x}\|} - \frac{g_j(x_k) - g_j(\bar{x})}{\|x_k - \bar{x}\|}$$
$$= (\nabla g_i(\bar{x}) - \nabla g_j(\bar{x}))' \left( \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \right) + \frac{o(\|x_k - \bar{x}\|)}{\|x_k - \bar{x}\|}.$$

It follows that

$$\frac{g_i(x_k) - g_j(x_k)}{\|x_k - \bar{x}\|} \to \nabla g_i(\bar{x})'d - \nabla g_j(\bar{x})'d < 0.$$

Consequently, $g_i(x_k) < g_j(x_k)$ for $k$ large enough which entails the contradiction $i \notin I(x_k) = I$. Now, (32) means that for all $i \in I$

$$\nabla g_i(\bar{x}) \in \arg\max \left\{ z'd \mid z \in \left\{ \nabla g_j(\bar{x}),\ j \in I(\bar{x}) \right\} \right\}$$
$$= \arg\max \left\{ z'd \mid z \in \operatorname{conv} \left\{ \nabla g_j(\bar{x}),\ j \in I(\bar{x}) \right\} \right\}$$
$$= \arg\max \left\{ z'd \mid z \in \partial g(\bar{x}) \right\} =: A.$$

Now, since $d \neq 0_n$, one has that $A \subseteq \operatorname{bd} \partial g(\bar{x})$. On the other hand, $A$ is convex by convexity of $\partial g(\bar{x})$. Therefore, the proven relation $\nabla g_i(\bar{x}) \in A$ for all $i \in I$ along with (31) imply the desired relation

$$u \in \operatorname{conv} \{\nabla g_i(\bar{x}),\ i \in I\} \subseteq A \subseteq \operatorname{bd} \partial g(\bar{x}).$$

The following paragraphs are devoted to establish the equality

$$\limsup_{x \to \overline{x},\ x \neq \overline{x}} \partial g(x) = \operatorname{bd}\partial g(\overline{x})$$

under the following condition: "for all supporting hyperplane $H$ to $\partial g(\bar{x})$, we have that $\{\nabla g_i(\bar{x}), i \in I(\bar{x})\} \cap H$ is affinely independent". So, we have to prove the remaining inclusion "$\supseteq$". To this aim, take any $u \in \operatorname{bd} \partial g(\bar{x})$ and let us show the existence of a sequence $(x_k) \subset \mathbb{R}^n$ converging to $\overline{x}$, with $x_k \neq \overline{x}$ for all $k$, such that

$$u \in \lim_{k \to \infty} \partial g(x_k).$$

Since $u \in \operatorname{bd} \partial g(\bar{x})$, there exists a supporting hyperplane $H$ to $\partial g(\bar{x})$ at $u$; so, we can write $H = \left\{ z \in \mathbb{R}^n : z'd = \delta \right\}$, with $0_n \neq d \in \mathbb{R}^n$, $\delta \in \mathbb{R}$,

$$u'd = \delta \text{ and } w'd \leq \delta,\ \forall w \in \partial g(\bar{x}) = \operatorname{conv} \{\nabla g_i(\bar{x}),\ i \in I(\overline{x})\}. \tag{33}$$

Let $I \subseteq I(\bar{x})$ be such that $\{\nabla g_i(\bar{x}),\ i \in I(\bar{x})\} \cap H = \{\nabla g_i(\bar{x}), i \in I\}$; in other words,

$$\nabla g_i(\overline{x})' d = \delta \ \text{ for all } \ i \in I, \text{ and } \nabla g_i(\overline{x})' d < \delta \ \text{ when } \ i \in I(\overline{x}) \setminus I. \tag{34}$$

Then, one easily checks that

$$u \in \operatorname{conv}\{\nabla g_i(\bar{x}),\ i \in I\}.$$

In fact, if we write

$$u = \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x})$$

for some $\lambda \in \mathbb{R}_+^{I(\bar{x})}$ with $\sum_{i \in I(\bar{x})} \lambda_i = 1$, we have

$$\delta = u'd = \sum_{i \in I} \lambda_i \nabla g_i(\bar{x})' d + \sum_{i \in I(\bar{x}) \setminus I} \lambda_i \nabla g_i(\bar{x})' d,$$

which implies $\lambda_i = 0$ for all $i \in I(\bar{x}) \setminus I$, as consequence of (34). By the current assumption, $\{\nabla g_i(\bar{x}),\ i \in I\}$ is affinely independent and, by simplicity, we may assume $I = \{1, \ldots, i_0\}$ ($i_0 \leq n$ since $\dim H = n - 1$). Then, from

$$g_i(\bar{x}) = g_1(\bar{x}), \text{ and } (\nabla g_i(\bar{x}) - \nabla g_1(\bar{x}))' d = 0, \quad \text{for all } i \in I,$$

by proceeding as in the proof of condition (i) above, we can establish the existence of a differentiable curve $\alpha$ such that

$$g_i(\alpha(t)) - g_1(\alpha(t)) = 0, \quad \text{whenever } -a < t < a \ (a > 0),$$

and

$$\alpha(0) = \bar{x},\ \dot{\alpha}(0) = d.$$

Again, let us consider a sequence of scalars $0 < t_k < a$, $k \in \mathbb{N}$, such that $t_k \to 0$ and define

$$x_k := \alpha(t_k), \quad \text{for all } k.$$

Since $\dot{\alpha}(0) = d \neq 0_n$, we may assume that $x_k \neq \bar{x}$ for all $k$. Then, as in the proof of condition (i), we have for $k$ large enough

$$\begin{aligned} &g_1(x_k) = \cdots = g_{i_0}(x_k), \\ &g_j(x_k) < g_1(x_k),\ j \in I(\bar{x}) \setminus \{1, \ldots, i_0\}, \end{aligned} \tag{35}$$

which yields

$$I(x_k) = I, \text{ and so } \partial g(x_k) = \operatorname{conv}\{\nabla g_i(x_k), i \in I\}.$$

Consequently,

$$u = \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) = \lim_{k \to \infty} \sum_{i \in I} \lambda_i \nabla g_i(x_k) \in \lim_{k \to \infty} \partial g(x_k),$$

which finishes the proof. □

*Remark 3.2* Theorem 3.1 in [1] shows that condition (ii) in the previous theorem also holds as equality in the case when function $g$ is convex, without differentiability assumptions, in which case, $\partial g$ represents the usual subdifferential of convex analysis.

*Remark 3.3* (i) Observe that under the linear independence constraint qualification (LICQ), all inclusions in Theorem 3.2 become equalities. Specifically, if $\{\nabla g_i(\overline{x})\,;\ i \in I(\overline{x})\}$ is linearly independent (and hence affinely independent), then the system $\{\nabla g_i(\overline{x})' d = 1;\ i \in I(\overline{x})\}$ obviously has a solution; in other words,

$$I(\overline{x}) \in \mathcal{D}_{AI}(\overline{x}).$$

Consequently, under LICQ,

$$\bigcup_{D \in \mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\} = \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in I(\overline{x})\}$$
$$= \operatorname{bdconv}\{\nabla g_i(\overline{x}),\ i \in I(\overline{x})\} = \operatorname{bd}\partial g(\overline{x}).$$

(ii) On the other hand, the well-known Mangasarian–Fromovitz constraints qualification (MFCQ) is not sufficient to guarantee equality in any of the inclusions in Theorem 3.2. For instance, if we consider $g(x) := \max\{\frac{1}{2}(x_1 + x_2), x_1 + x_2, x_1, x_1 - x_2\}$, and take $\overline{x} = (0, 0)$, it is immediate that

$$\begin{aligned}
\{(1,1),(1,-1)\} &= \bigcup_{D \in \mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\} \\
&\subsetneqq \limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x) = \operatorname{conv}\{(1,1),(1,-1)\} \\
&\subsetneqq \bigcup_{D \in \mathcal{D}(\overline{x}) \cup \mathcal{D}^0(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\} \\
&= \operatorname{conv}\{(1,1),(1,-1)\} \cup \operatorname{conv}\{(1,1),(\tfrac{1}{2},\tfrac{1}{2})\} \\
&\subsetneqq \operatorname{bd}\partial g(\overline{x}) = \operatorname{bdconv}\{(1,1),(1,-1),(\tfrac{1}{2},\tfrac{1}{2})\}.
\end{aligned}$$

### 3.3 Application to the Calmness Modulus of $\mathcal{C}^1$-Systems

As immediate consequence of Theorems 2.1 and 3.2, we obtain the following result which provides an upper bound for the calmness modulus of $\mathcal{C}^1$-systems and, in addition, a lower bound in the case of $\mathcal{C}^1$-convex systems.

**Corollary 3.1** *Let us consider the $\mathcal{C}^1$-system introduced in (3). Let $\mathcal{F}$ be the associated feasible set mapping, i.e.,*

$$\mathcal{F}(b) := \left\{x \in \mathbb{R}^n :\ f_i(x) \le b_i, \quad \text{for all } i = 1, \ldots, m\right\}.$$

*Consider a given $(\overline{b}, \overline{x}) \in \operatorname{gph}\mathcal{F}$ such that $g(\overline{x}) = 0$, where $g$ is the max-function (5), i.e.,*

$$g(x) := \max_{i=1,\ldots,m} \{f_i(x) - \overline{b}_i\}, \quad x \in \mathbb{R}^n.$$

*Then*:

(i) *We have*

$$\text{clm}\mathcal{F}(\overline{b}, \overline{x}) \leq \left( \min_{D \in \mathcal{D}(\overline{x}) \cup \mathcal{D}^0(\overline{x})} d_* \left(0_n, \text{conv}\left\{\nabla f_i\left(\overline{x}\right), \ i \in D\right\}\right) \right)^{-1}$$
$$\leq \left(d_* \left(0_n, \text{bd}\partial g\left(\overline{x}\right)\right)\right)^{-1};$$

(ii) *If, additionally, the functions* $f_i$*,* $i = 1, \ldots, m$*, are convex, then*

$$\left( \min_{D \in \mathcal{D}_{AI}(\overline{x})} d_* \left(0_n, \text{conv}\left\{\nabla f_i\left(\overline{x}\right), \ i \in D\right\}\right) \right)^{-1} \leq \text{clm}\mathcal{F}(\overline{b}, \overline{x}).$$

*Proof* (i) The proof comes straightforwardly from Theorems 2.1(i) and 3.2(i). (ii) It is a direct consequence of Theorems 2.1(ii) and 3.2(i). □

*Remark 3.4* (i) According to Remark 3.3(i) and Theorem 2.1(ii), for any $\mathcal{C}^1$ convex system verifying LICQ at $(\overline{b}, \overline{x})$, all inequalities in the previous corollary become equalities.

(ii) On the other hand, all inequalities of the previous corollary may be strict. Specifically, in the example given in Remark 3.3(ii), with $\mathbb{R}^2$ endowed with the Euclidean norm, one has

$$\text{clm}\mathcal{F}\left(\overline{b}, \overline{x}\right) = 1,$$
$$\left( \min_{D \in \mathcal{D}_{AI}(\overline{x})} d_* \left(0_n, \text{conv}\left\{\nabla f_i\left(\overline{x}\right), \ i \in D\right\}\right) \right)^{-1} = \frac{1}{\sqrt{2}},$$
$$\left( \min_{D \in \mathcal{D}(\overline{x}) \cup \mathcal{D}^0(\overline{x})} d_* \left(0_n, \text{conv}\left\{\nabla f_i\left(\overline{x}\right), \ i \in D\right\}\right) \right)^{-1} = \sqrt{2},$$
$$\left(d_* \left(0_n, \text{bd}\partial g\left(\overline{x}\right)\right)\right)^{-1} = \frac{\sqrt{10}}{2}.$$

In the following example, inequalities in Corollary 3.1(i) are strict, whereas the lower bound in Corollary 3.1(ii) is attained.

*Example 3.2* Consider the system, in $\mathbb{R}^2$ with the Euclidean norm,

$$\sigma(b) := \left\{ \begin{array}{l} g_1(x) := 2x_1^2 + x_2^2 + 4x_1 + 2x_2 \leq b_1, \\ g_2(x) := x_1^2 + x_2^2 - 4x_1 \leq b_2, \\ g_3(x) := -\frac{1}{2}x_1 \leq b_3, \end{array} \right\}$$

and take $\overline{b} = 0_3$ and $\overline{x} = 0_2$. Now, the associated max-function $g : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$g(x) = \max\{g_1(x), \ g_2(x), \ g_3(x)\}.$$

In this case, we easily see that:

$$\bigcup_{D\in\mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\left\{\nabla g_i\left(\overline{x}\right),\ i\in D\right\} = \operatorname{conv}\left\{(-4,0),(4,2)\right\}. \tag{36}$$

Moreover, let us see that

$$\limsup_{x\to\overline{x},\ g(x)>0} \partial g(x) = \operatorname{conv}\left\{(-4,0),(4,2)\right\}.$$

Observe that $g(x)>0$ implies $g(x)=\max\{g_1(x),g_2(x)\}>g_3(x)$. In fact, one can easily check that $g_2(x)\le g_3(x)$ yields $g_3(x)\le 0$ and, then, if simultaneously $g_1(x), g_2(x)\le g_3(x)$, we have $g(x)\le 0$. As a consequence of that,

$$\begin{aligned}\limsup_{x\to\overline{x},\ g(x)>0} \partial g(x) &\subseteq \limsup_{x\to\overline{x},\ g(x)>g_3(x)} \partial g(x)\\ &\subseteq \lim_{x\to\overline{x}} \operatorname{conv}\left\{\nabla g_1(x),\nabla g_2(x)\right\} = \operatorname{conv}(-4,0),(4,2)\}.\end{aligned}$$

Then, by the first inclusion in Theorem 3.2(i) and (36),

$$\operatorname{clm}\mathcal{F}(\overline{b},\overline{x}) = \left[d_*\left(0_n,\operatorname{conv}\left\{(-4,0),(4,2)\right\}\right)\right]^{-1} = \frac{\sqrt{17}}{4}.$$

However,

$$\begin{aligned}&\bigcup_{D\in\mathcal{D}(\overline{x})\cup\mathcal{D}^0(\overline{x})} \operatorname{conv}\left\{\nabla g_i\left(\overline{x}\right),\ i\in D\right\}\\ &= \operatorname{conv}(-4,0),(4,2)\}\cup\operatorname{conv}(-4,0),(-\tfrac{1}{2},0)\}\\ &\subsetneqq \operatorname{bd}\partial g\left(\overline{x}\right) = \operatorname{bd}\operatorname{conv}\left\{(-4,0),(4,2),\left(-\tfrac{1}{2},0\right)\right\},\end{aligned}$$

and the reader can immediately check that both inequalities in Corollary 3.1(i) are strict.

*Remark 3.5* Let us remark that in the last two examples, if we approach $\overline{x}$ by directional sequences, i.e., sequences of the type $x_k=\overline{x}+t_k u$ with $u\neq 0_2$ and $t_k\to 0$, and we represent this directional convergence with the symbol $x\overset{d}{\to}\overline{x}$, we shall obtain only the extreme points of the sets generated by arbitrary convergent sequences. More specifically, in Example 3.2,

$$\limsup_{x\overset{d}{\to}\overline{x},\ g(x)>0} \partial g(x) = \left\{(-4,0),(4,2)\right\}.$$

This observation corresponds to the statement in Theorem 6.3.6 in [24].

The following example illustrates a situation in which calmness fails, i.e., $\operatorname{clm}\mathcal{F}(\overline{b},\overline{x})=+\infty$, and the lower bound in Corollary 3.1(ii) is finite.

*Example 3.3* Consider the system

$$\sigma(b) := \left\{ \begin{array}{l} x_1^2 + x_2^2 \le b_1, \\ x_1 + x_2 \le b_2, \end{array} \right\}$$

and take $\overline{b} = 0_2$ and $\overline{x} = 0_2$.

In this case,

$$\bigcup_{D \in \mathcal{D}_{AI}(\overline{x})} \text{conv} \{\nabla g_i (\overline{x}), \ i \in D\} = \{(1, 1)\}.$$

However, one obtains from [1, Theorem 3.1] (observing that $g(x) > 0 \Leftrightarrow x \neq \overline{x}$; see Remark 3.2)

$$\limsup_{x \to \overline{x}, \ g(x) > 0} \partial g(x) = \limsup_{x \to \overline{x}, \ x \neq \overline{x}} \partial g(x) = \text{bd} \partial g(\overline{x}) = \text{conv} \{(0, 0), (1, 1)\},$$

and then $\text{clm}\mathcal{F}(\overline{b}, \overline{x}) = +\infty$.

*Remark 3.6* (i) If we modify $g_1$ in the previous example by adding $\frac{1}{2}(x_1 + x_2)$, i.e.,

$$g_1(x) := x_1^2 + x_2^2 + \frac{1}{2}(x_1 + x_2),$$

then one still has $\mathcal{D}_{AI}(\overline{x}) = \{\{2\}\}$ and, by approaching $0_2$ by points of the circumference $x_1^2 + x_2^2 = \frac{1}{2}(x_1 + x_2)$ different from $0_2$ one checks

$$\limsup_{x \to \overline{x}, \ g(x) > 0} \partial g(x) = \text{bd} \partial g(\overline{x}) = \text{conv} \left\{ (\tfrac{1}{2}, \tfrac{1}{2}), (1, 1) \right\},$$

and then $\text{clm}\mathcal{F}(\overline{b}, \overline{x}) = \left\| (\tfrac{1}{2}, \tfrac{1}{2}) \right\|_*^{-1}$.

(ii) If we replace this modified system by its first-order approach at $0_2$, i.e., we consider $g_1(x) = \frac{1}{2}(x_1 + x_2)$ by removing the second-order term, then $\text{clm}\mathcal{F}(\overline{b}, \overline{x}) = \|(1, 1)\|_*^{-1}$.

We finish this subsection paying attention to the particular case when $\mathcal{F}$ is isolatedly calm at $(\overline{b}, \overline{x})$, i.e., when together with calmness we have the uniqueness assumption $\mathcal{F}(\overline{b}) = \{\overline{x}\}$. Firstly, we show that the specification of Theorem 3.2(i) to this case reads in a simpler way, as far as $\mathcal{D}^0(\overline{x})$ can be removed of the statement. Secondly, it is well known from variational analysis (see, for example, [7]) that if $\mathcal{F}$ is isolatedly calm at $(\bar{b}, \bar{x})$, then its calmness modulus equals the outer norm of its graphical derivative, $D\mathcal{F}$; formally,

$$\text{clm}\, \mathcal{F}(\bar{b}, \bar{x}) = \sup \left\{ \|w\| \,|\, w \in D\mathcal{F}(\bar{b}, \bar{x})(u), \ \|u\| \le 1 \right\}. \tag{37}$$

Recall that, by definition,

$$w \in D\mathcal{F}(\bar{b}, \bar{x})(u) \Longleftrightarrow (u, w) \in T_{\text{gph}\, \mathcal{F}}(\bar{b}, \bar{x}),$$

where $T_{\text{gph}\, \mathcal{F}}$ refers to the contingent cone of the graph of $\mathcal{F}$.

**Theorem 3.3** *Let $\mathcal{F}$ be the feasible set mapping* (4) *and $g$ the max-function* (5). *Assume that $\mathcal{F}$ is isolatedly calm at $(\overline{b}, \overline{x})$. Then:*

(i) $\limsup_{x \to \overline{x},\ g(x) > g(\overline{x})} \partial g(x) \subseteq \bigcup_{D \in \mathcal{D}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\}$

(ii) $\operatorname{clm} \mathcal{F}(\bar{b}, \bar{x}) = \sup\{\|w\| : \|u\|_\infty \leq 1,\ \langle \nabla f_i(\bar{x}), w\rangle \leq u_i \ \ (i \in I(\bar{x}))\}$.

*Proof* (i) The proof follows the same argument as in the second part of the proof of Theorem 3.2(i), just observing that under the current isolated calmness assumption the case $\alpha = 0$ cannot occur. This fact allows us to remove $\mathcal{D}^0(\overline{x})$ in the statement of that theorem. Specifically, keeping the notation of that proof, from (30) we can write

$$0 < \frac{g(x_k) - g(\overline{x})}{\|x_k - \bar{x}\|} \to \alpha.$$

Then, we have $\alpha > 0$ as consequence of

$$\lim_{k\to\infty} \frac{\|x_k - \bar{x}\|}{g(x_k) - g(\overline{x})} = \lim_{k\to\infty} \frac{d(x_k, \mathcal{F}(\bar{b}, \bar{x}))}{g(x_k) - g(\overline{x})} \leq \operatorname{clm} \mathcal{F}(\bar{b}, \bar{x}) < +\infty.$$

(ii) Define the mapping $H(b, x) := f(x) - b \equiv (f_i(x) - b_i)_{1 \leq i \leq m}$ and observe that $\operatorname{gph} \mathcal{F} = H^{-1}\left(\mathbb{R}^m_-\right)$. Moreover, $\nabla H(\bar{b}, \bar{x}) = \left(-I | \nabla f(\bar{x})'\right)$ is surjective. Then, (see [10]),

$$T_{\operatorname{gph} \mathcal{F}}(\bar{b}, \bar{x}) = \left[\nabla H(\bar{b}, \bar{x})\right]^{-1} T_{\mathbb{R}^m_-}(H(\bar{b}, \bar{x})),$$

where the upper index '$-1$' refers to the set-valued inverse. Finally, it is well known that

$$T_{\mathbb{R}^m_-}(H(\bar{b}, \bar{x})) = \{y | y_i \leq 0;\ i \in I(\bar{x})\}.$$

It follows from the structure of $\nabla H(\bar{b}, \bar{x})$ that

$$(u, w) \in T_{\operatorname{gph} \mathcal{F}}(\bar{b}, \bar{x}) \Longleftrightarrow -u + \nabla f(\bar{x})' w \in T_{\mathbb{R}^m_-}(H(\bar{b}, \bar{x})).$$

In other words,

$$(u, w) \in T_{\operatorname{gph} \mathcal{F}}(\bar{b}, \bar{x}) \Longleftrightarrow \langle \nabla f_i(\bar{x}), w\rangle \leq u_i \ \ (i \in I(\bar{x})),$$

which in combination with (37) yields the asserted formula. □

## 4 Computing the Calmness Modulus of the Argmin Mapping for Linear Programs

In this section, a suitable Karush–Kuhn–Tucker (KKT) index set approach will allow us to derive the exact calmness modulus of $\mathcal{S}$, defined in (12), at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \operatorname{gph}\mathcal{S}$

under non-uniqueness assumptions, i.e., without assuming $\mathcal{S}(\overline{c}, \overline{b}) = \{\overline{x}\}$. In our way to prove this result, we have to extend the lower bound and sharpen the upper bound given, respectively, in [3, Theorems 6 and 7] (see Sect. 2 for more details).

To start with, the next example shows that inequality in (15) may be strict when $\mathcal{S}\left(\overline{c}, \overline{b}\right)$ is not a singleton (see Remark 2.2) and gives a hint to sharpen such an upper bound. In Corollary 4.1, we will see that this sharpened upper bound is in fact the exact calmness modulus of $\mathcal{S}$.

*Example 4.1* Consider the nominal problem (in $\mathbb{R}^2$ endowed with the Euclidean norm)

$$\begin{array}{lll} P\left(\overline{c}, \overline{b}\right): & Min\ x_1 & \\ s.t. & -x_1 \leq 0, & (i=1), \\ & -x_2 \leq 0, & (i=2), \\ & -x_1 - x_2 \leq 0, & (i=3). \end{array}$$

Let $\overline{x} := 0_2$. By appealing to Theorem 2.2, applied to mappings $\mathcal{L}_D$—which are nothing else but feasible set mappings associated with enlarged systems—at $\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right) \in \text{gph}\, \mathcal{L}_D$ with $D \in \mathcal{K}_{\overline{c},\overline{b}}\left(\overline{x}\right)$, we obtain the following table:

| $D \in \mathcal{K}_{\overline{c},\overline{b}}\left(\overline{x}\right)$ | $\text{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right)$ |
|---|---|
| $\{1\},\ \{1,2\}$ | $\sqrt{2}$ |
| $\{1,3\}$ | $\sqrt{5}$ |

Now Theorem 2.3 ensures $\text{clm}\mathcal{S}(\left(\overline{c}, \overline{b}\right), \overline{x}) \leq \sqrt{5}$.

An *ad hoc* geometrical argument could show that $\text{clm}\mathcal{S}(\left(\overline{c}, \overline{b}\right), \overline{x}) = \sqrt{2}$ in the previous example. The underlying idea is that those $D \in \mathcal{K}_{\overline{c},\overline{b}}\left(\overline{x}\right)$ with some zero KKT multiplier $\lambda_i$ in an expression $-\overline{c} = \sum_{i \in D} \lambda_i a_i$ are not relevant. In other words, the key fact consists of confining ourselves to those KKT subsets which are minimal with respect to the inclusion order, and consequently the associated multipliers are all of them nonzero. Accordingly, we consider, associated with $(\left(\overline{c}, \overline{b}\right), \overline{x}) \in \text{gph}\mathcal{S}$, the family of *minimal KKT subsets* given by

$$\mathcal{M}_{\overline{c},\overline{b}}\left(\overline{x}\right) = \left\{D \in \mathcal{K}_{\overline{c},\overline{b}}\left(\overline{x}\right) : D \text{ is minimal for the inclusion order}\right\}.$$

Observe that in the previous example one has $\mathcal{M}_{\overline{c},\overline{b}}\left(\overline{x}\right) = \{\{1\}\}$.

*Remark 4.1* In the special case $\overline{c} = 0_n$, it is easy to see (thanks to Theorem 2.3) that

$$\text{clm}\mathcal{S}(\left(\overline{c}, \overline{b}\right), \overline{x}) = \text{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x}) = \text{clm}\mathcal{F}(\overline{b}, \overline{x}),$$

and we already have an expression for the latter. So, in the sequel we could assume $\overline{c} \neq 0_n$. Nevertheless, the case $\overline{c} = 0_n$ is also included in our results if we use the convention $\mathcal{M}_{\overline{c},\overline{b}}\left(\overline{x}\right) = \{\emptyset\}$ whenever $\overline{c} = 0_n$, and $\mathcal{L}_\emptyset := \mathcal{F}$.

**Theorem 4.1** *Let* $\left(\overline{c}, \overline{b}\right) \in \mathbb{R}^n \times \mathbb{R}^m$*, and assume* $\overline{x} \in \mathcal{S}\left(\overline{c}, \overline{b}\right)$*. Then*

$$\operatorname{clm}\mathcal{S}(\left(\overline{c}, \overline{b}\right), \overline{x}) \leq \max_{D \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right). \tag{38}$$

*Proof* Under the current hypotheses, Theorem 2.3 establishes

$$\operatorname{clm}\mathcal{S}(\left(\overline{c}, \overline{b}\right), \overline{x}) = \operatorname{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x}),$$

where $\mathcal{S}_{\overline{c}} := \mathcal{S}\left(\overline{c}, \cdot\right)$, i.e., $\mathcal{S}_{\overline{c}}(b) = \mathcal{S}\left(\overline{c}, b\right)$ for each $b \in \mathbb{R}^m$. Let us write

$$\operatorname{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x}) = \lim_{r \to \infty} \frac{d\left(x^r, \mathcal{S}\left(\overline{c}, \overline{b}\right)\right)}{\left\|b^r - \overline{b}\right\|_\infty} \tag{39}$$

for some $\mathbb{R}^m \ni b^r \to \overline{b}$ (with $b^r \neq \overline{b}$ for all $r \in \mathbb{N}$) and some $\mathcal{S}_{\overline{c}}(b^r) \ni x^r \to \overline{x}$. According to the KKT conditions, take for each $r$ a certain $D_r \subseteq T_{b^r}\left(x^r\right)$ with $|D_r| \leq n$ (because of Carathéodory's Theorem) such that

$$-\overline{c} \in \operatorname{cone}\left\{a_i,\ i \in D_r\right\}. \tag{40}$$

The finiteness of $\{1, \ldots, m\}$ enables us assume for a suitable subsequence (denoted as the whole sequence for simplicity) that $D_r = D$ (independent of $r$). Then, it is clear that, for such a subsequence, in (40) we may assume that all KKT multipliers are nonzero and that set $D$ is minimal with this property. Moreover, $D \subseteq T_{b^r}\left(x^r\right)$ for all $r$ clearly implies $D \subseteq T_{\overline{b}}\left(\overline{x}\right)$ by just taking limits in $a_i' x^r = b_i^r$ for each $i \in D$. Accordingly, we can write

$$D \in \mathcal{M}_{\overline{c},\overline{b}}\left(\overline{x}\right).$$

Since, on the one hand, $D \subseteq T_{b^r}\left(x^r\right)$ clearly implies $x^r \in \mathcal{L}_D\left(b^r, -b_D^r\right)$ and, on the other hand, $\mathcal{L}_D\left(\overline{b}, -\overline{b}_D\right) \subseteq \mathcal{S}\left(\overline{c}, \overline{b}\right)$ (i.e., every KKT point is optimal), (39) entails, taking into account the obvious fact that

$$\left\|\left(b^r, b_D^r\right) - \left(\overline{b}, -\overline{b}_D\right)\right\|_\infty = \left\|b^r - \overline{b}\right\|_\infty$$

(the first one in $\mathbb{R}^m \times \mathbb{R}^D$ and the second in $\mathbb{R}^m$),

$$\operatorname{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x}) \leq \limsup_{r \to \infty} \frac{d\left(x^r, \mathcal{L}_D\left(\overline{b}, -\overline{b}_D\right)\right)}{\left\|\left(b^r, b_D^r\right) - \left(\overline{b}, -\overline{b}_D\right)\right\|_\infty} \leq \operatorname{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right).$$

□

Next, we are going to see that the right-hand side of (38) already stands as a lower bound on $\operatorname{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x})$ (and hence on $\operatorname{clm}\mathcal{S}(\left(\overline{c}, \overline{b}\right), \overline{x})$) in the following semi-infinite setting, which obviously includes the case when $T$ is finite:

- $T$ is a compact Hausdorff space,
- The given function $a \equiv (a_t)_{t\in T}$ belongs to $C(T, \mathbb{R}^n)$,
- Parameter $b \equiv (b_t)_{t\in T}$ belongs to $C(T, \mathbb{R})$,
- The optimal set mapping $\mathcal{S} : \mathbb{R}^n \times C(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$ is defined by

$$\mathcal{S}(c, b) := \arg\min\{c'x : a_t'x \leq b_t,\ t \in T\},$$

which is a natural extension of (12). The rest of notation (sets $\mathcal{K}_{\overline{c},\overline{b}}(\overline{x})$ and $\mathcal{M}_{\overline{c},\overline{b}}(\overline{x})$, and mappings $\mathcal{S}_{\overline{c}}$ and $\mathcal{L}_D$) also remains unchanged, but adapted to the new setting.

Hereafter, in this section, let us assume the previous framework. Theorem 6 in [3] shows that the last term in (15), i.e., $\max_{D\in\mathcal{K}_{\overline{c},\overline{b}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right)$, is a lower bound on $\operatorname{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x})$ in this new setting when we also assume: (i) $\mathcal{S}(\overline{c}, \overline{b}) = \{\overline{x}\}$, (ii) the Slater constraint qualification at the nominal parameter $\overline{b}$ (i.e., the existence of some $\widehat{x} \in \mathbb{R}^n$ such that $a_t'\widehat{x} < \overline{b}_t$ for all $t \in T$). In Theorem 4.2 below, we show that the (possibly) sharper upper bound $\max_{D\in\mathcal{M}_{\overline{c},\overline{b}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right)$ also stands as a lower bound without assuming neither (i) nor (ii).

For any $D \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x})$, we consider the supremum function, $f_D : \mathbb{R}^n \to \mathbb{R}$ given by

$$\begin{aligned} f_D(x) &:= \sup\left\{\langle a_t, x\rangle - \bar{b}_t,\ t \in T;\ -\langle a_t, x\rangle + \bar{b}_t,\ t \in D\right\} \\ &= \sup\left\{\langle a_t, x\rangle - \bar{b}_t,\ t \in T\backslash D;\ \left|\langle a_t, x\rangle - \bar{b}_t\right|,\ t \in D\right\}, \end{aligned}$$

Observe that

$$\mathcal{L}_D\left(\bar{b}, -\bar{b}_D\right) = [f_D = 0] \subseteq \mathcal{S}\left(\bar{c}, \bar{b}\right) \text{ for all } D \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x}). \tag{41}$$

Let us also observe that, as a direct consequence of Theorem 2.1,

$$\operatorname{clm}\mathcal{L}_D\left(\left(\bar{b}, -\bar{b}_D\right), \overline{x}\right) = \limsup_{\substack{x\to\overline{x} \\ f_D(x)>0}} \frac{1}{d_*\left(0_n, \partial f_D(x)\right)}. \tag{42}$$

**Proposition 4.1** *In our current semi-infinite setting, let* $\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) \in \operatorname{gph}\mathcal{S}$. *Then*

$$\mathcal{L}_D\left(\overline{b}, -\overline{b}_D\right) = \mathcal{S}\left(\bar{c}, \bar{b}\right), \quad \textit{for all } D \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x}).$$

*Proof* We only have to prove the inclusion "$\supseteq$" [recall (41)]. Reasoning by contradiction, assume the existence of a certain $\overline{D} \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x})$ and some $\widehat{x} \in \mathcal{S}\left(\bar{c}, \bar{b}\right)\backslash\mathcal{L}_{\overline{D}}\left(\overline{b}, -\overline{b}_{\overline{D}}\right)$. Observe that, since $\widehat{x}$ is feasible for $P\left(\bar{c}, \bar{b}\right)$, we have

$$a_t'(\overline{x} - \widehat{x}) = \overline{b}_t - a_t'\widehat{x} \geq 0, \quad \text{for all } t \in \overline{D}, \tag{43}$$

while condition $\widehat{x} \notin \mathcal{L}_{\overline{D}}\left(\overline{b}, -\overline{b}_{\overline{D}}\right)$ yields

$$a_t'(\overline{x} - \widehat{x}) = \overline{b}_t - a_t'\widehat{x} > 0, \quad \text{for some } t \in \overline{D}. \tag{44}$$

Moreover, $\overline{D} \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x})$ entails the existence of a $\lambda_t > 0$, for each $t \in \overline{D}$ such that

$$-\bar{c} = \sum_{t \in \overline{D}} \lambda_t a_t.$$

Then

$$-\bar{c}'(\overline{x} - \widehat{x}) = \sum_{t \in \overline{D}} \lambda_t a_t'(\overline{x} - \widehat{x}).$$

Observe that $\bar{c}'(\overline{x} - \widehat{x}) = 0$ (since $\overline{x}, \widehat{x} \in \mathcal{S}\left(\bar{c}, \bar{b}\right)$), which, according to (43), yields

$$\lambda_t a_t'(\overline{x} - \widehat{x}) = 0, \quad \text{for all } t \in \overline{D}.$$

Then, applying (44) we attain the contradiction (with the minimality condition of $\overline{D}$)

$$\lambda_t = 0, \quad \text{for some } t \in \overline{D}.$$

□

**Theorem 4.2** *In our current semi-infinite setting, let* $\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) \in \mathrm{gph}\mathcal{S}$. *Then*

$$\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) \geq \mathrm{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) \geq \sup_{D \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x})} \mathrm{clm}\mathcal{L}_D\left(\left(\bar{b}, -\bar{b}_D\right), \overline{x}\right).$$

*Proof* The first inequality comes directly from the definition of calmness modulus. Now, we are going to prove the second inequality in the non-trivial case $\bar{c} \neq 0_n$ (see Remark 4.1). Take any $\overline{D} \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x})$ and let us see that $\mathrm{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) \geq \mathrm{clm}\mathcal{L}_{\overline{D}}\left(\left(\bar{b}, -\bar{b}_{\overline{D}}\right), \overline{x}\right)$. From (42), we can write

$$\mathrm{clm}\mathcal{L}_{\overline{D}}\left(\left(\bar{b}, -\bar{b}_{\overline{D}}\right), \overline{x}\right) = \lim_{r \to +\infty} \frac{1}{\|u^r\|_*},$$

for a certain sequence $\{u^r\}_{r \in \mathbb{N}}$ verifying $u^r \in \partial f_{\overline{D}}(x^r)$, for all $r$, where $\{x^r\}_{r \in \mathbb{N}}$ is such that

$$\lim_{r \to +\infty} x^r = \overline{x} \text{ and } f_{\overline{D}}\left(x^r\right) > 0, \quad \text{for all } r.$$

In particular, $x^r \notin \mathcal{L}_{\overline{D}}\left(\bar{b}, -\bar{b}_D\right)$, for all $r$, and then, applying Proposition 4.1,

$$x^r \notin \mathcal{S}_{\bar{c}}\left(\bar{b}\right), \quad \text{for all } r.$$

For each $r$, let $\widetilde{x}^r \in \mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right)$ a best approximation of $x^r$ in $\mathcal{S}_{\bar{c}}\left(\bar{b}\right)$, i.e.,

$$\left\|x^r - \widetilde{x}^r\right\| = d\left(x^r, \mathcal{S}_{\bar{c}}(\bar{b})\right), \quad \text{for all } r.$$

We have, for each $r$,

$$\left\|x^r - \widetilde{x}^r\right\| \left\|u^r\right\|_* \geq (u^r)'(x^r - \widetilde{x}^r) \geq f_{\overline{D}}\left(x^r\right) - f_{\overline{D}}\left(\widetilde{x}^r\right) = f_{\overline{D}}\left(x^r\right),$$

where we have appealed again to the previous proposition to ensure that $f_{\overline{D}}\left(\widetilde{x}^r\right) = 0$, for all $r$. Consequently,

$$\left\|x^r - \widetilde{x}^r\right\| \geq \frac{f_{\overline{D}}\left(x^r\right)}{\|u^r\|_*}, \quad \text{for all } r. \tag{45}$$

Now, following the same argument as is the last part of the proof of [3, Theorem 6] (just by adapting the notation), we may construct a sequence $\{b^r\} \subset C\left(T, \mathbb{R}\right)$ such that

$$x^r \in \mathcal{S}_{\bar{c}}\left(b^r\right) \text{ and } \left\|b^r - \bar{b}\right\|_\infty \leq \left(1 + \frac{1}{r}\right) f_{\overline{D}}\left(x^r\right), \quad \text{for all } r. \tag{46}$$

Just for completeness, at this moment we write the definition of $b^r$. For each $r$,

$$b_t^r := \left(1 - \varphi_r\left(t\right)\right) a_t' x^r + \varphi_r\left(t\right)\left(\bar{b}_t + f_{\overline{D}}\left(x^r\right)\right),$$

where $\varphi_r\left(t\right)$ is a continuous function from $T$ to $[0, 1]$ such that

$$\varphi_r\left(t\right) = \begin{cases} 0 & \text{if } t \in \overline{D}, \\ 1 & \text{if } a_t' x^r - \bar{b}_t \leq -\left(1 + \frac{1}{r}\right) f_{\overline{D}}\left(x^r\right), \end{cases}$$

whose existence is guaranteed by Urysohn's lemma. Finally, taking (45) and (46) into account, we obtain the aimed inequality

$$\begin{aligned} \operatorname{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) &\geq \lim_{r\to\infty} \frac{\left\|x^r - \widetilde{x}^r\right\|}{\left\|b^r - \bar{b}\right\|_\infty} \geq \lim_{r\to\infty}\left(1 + \frac{1}{r}\right)^{-1} \left\|u^r\right\|_*^{-1} \\ &= \operatorname{clm}\mathcal{L}_{\overline{D}}\left(\left(\bar{b}, -\bar{b}_D\right), \overline{x}\right). \end{aligned}$$

□

**Corollary 4.1** *Assume that $T$ is finite and let $\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) \in \operatorname{gph}\mathcal{S}$. Then*

$$\operatorname{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) = \operatorname{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) = \max_{D \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left(\left(\bar{b}, -\bar{b}_D\right), \overline{x}\right).$$

## 5 Perspectives

Concerning possible perspectives for future research, we point out the fact that in our analysis of outer subdifferentials and calmness moduli, we are looking for conceptually tractable expressions (exact formulae or estimations), in the sense that they only involve the nominal data (nominal point and nominal parameter). In this line, there are at least three open problems to tackle: first, the possibility to sharpen some of our estimations (subsets and supersets) for the outer limit of subdifferentials, with the associated repercussions on the calmness modulus of the feasible set mapping; second, the possibility to extend our analysis of the calmness modulus of the argmin mapping to nonlinear problems; and third, the analysis of semi-infinite systems and problems, starting with the linear case.

More specifically, in relation to the first problem (sharper estimations), it would be of interest to explore the possibility of replacing $\mathcal{D}_{AI}(\overline{x})$ with $\mathcal{D}(\overline{x})$ in the first term of the chain of inclusions provided in Theorem 3.2(i). Of course the affine independence assumption is essential in the current proof; so that the inclusion

$$\bigcup_{D\in\mathcal{D}(\overline{x})} \operatorname{conv}\left\{\nabla g_i\left(\overline{x}\right),\ i\in D\right\} \subseteq \limsup_{x\to\overline{x},\ g(x)>g(\overline{x})} \partial g(x)$$

remains as an open problem.

In relation to the second problem (calmness modulus of the argmin mapping for nonlinear problems), we would like to mention the gap which occurs when passing from linear to nonlinear problems pointed out in [25, Section 3], where a characterization of the calmness property of the argmin mapping becomes just a sufficient condition when replacing a linear function (either in the objective function or in the constraints) with a convex quadratic one.

Finally, in relation to the third problem (the semi-infinite setting), we point out that linear semi-infinite systems naturally appear when we apply the standard linearization technique (by means of the Fenchel–Legendre conjugate) to (finite or semi-infinite) convex inequality systems. Equality in Theorem 4.2 for linear semi-infinite programs remains as an open problem.

## 6 Conclusions

The main contributions of this work are developed in two different directions: the analysis of certain outer limits of subdifferentials of max-functions under different assumptions, and the computation of the calmness moduli for certain feasible and optimal set mappings. We point out the fact that the two different kind of results have a common starting point: the background about the calmness modulus of feasible set mappings associated with linear inequality systems.

With respect to the outer limit of subdifferentials, we deal with two different situations depending on the way of approaching the nominal point: either through points which are different from it or through points where the function under consideration takes greater values than the nominal value. The first case was already dealt in [1] for a convex function (without differentiability assumptions), and the new results provided in this paper constitute its (nonconvex) differentiable counterpart. The second type of outer limits is the one which constitutes a key ingredient in the estimations of the calmness modulus of feasible set mappings associated with right-hand-side perturbations of a nominal system (recall Theorem 2.1) under differentiability/convexity assumptions. So, advances in the knowledge of this second outer limit yield advances in the knowledge of the calmness modulus of the feasible set mapping. In relation to this point, with the aim of developing practical implementations of this calmness modulus, we point out the importance of obtaining formulae which only depend on the nominal elements. As it is specifically shown in the paper, the polyhedral case, corresponding to linear systems under right-hand-side perturbations, admits a particularly tractable formulation.

**Acknowledgments** This research has been partially supported by Grants MTM2011-29064-C03 (02-03) and MTM2014-59179-C2-(1,2)-P from MINECO, Spain.

## References


1. Cánovas, M.J., Hantoute, A., Parra, J., Toledo, F.J.: Boundary of subdifferentials and calmness moduli in linear semi-infinite optimization. Optim. Lett. **9**, 513–521 (2015)
2. Cánovas, M.J., Hantoute, A., Parra, J., Toledo, F.J.: Calmness of fully perturbed linear programs. Math. Program. Ser. A (2015). doi:10.1007/s10107-015-0926-x
3. Cánovas, M.J., Kruger, A.Y., López, M.A., Parra, J., Thera, M.A.: Calmness modulus of linear semi-infinite programs. SIAM J. Optim. **24**, 29–48 (2014)
4. Cánovas, M.J., López, M.A., Parra, J., Toledo, F.J.: Calmness of the feasible set mapping for linear inequality systems. Set Valued Var. Anal. **22**, 375–389 (2014)
5. Robinson, S.M.: Some continuity properties of polyhedral multifunctions. Mathematical programming at Oberwolfach (Proc. Conf., Math. Forschungsinstitut, Oberwolfach, 1979). Math. Program. Stud. **14**, 206–214 (1981)
6. Robinson, S.M.: A characterization of stability in linear programming. Oper. Res. **25**, 435–447 (1977)
7. Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings: A View from Variational Analysis. Springer, New York (2009)
8. Klatte, D. Kummer, B.: Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications. Nonconvex Optim. Appl. 60. Kluwer Academic, Dordrecht (2002)
9. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory. Springer, Berlin (2006)
10. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Berlin (1998)
11. Gfrerer, H.: First order and second order characterizations of metric subregularity and calmness of constraint set mappings. SIAM J. Optim. **21**, 1439–1474 (2011)
12. Henrion, R., Outrata, J.: Calmness of constraint systems with applications. Math. Program. B **104**, 437–464 (2005)
13. Klatte, D., Kummer, B.: Optimization methods and stability of inclusions in Banach spaces. Math. Program. B **117**, 305–330 (2009)
14. Klatte, D., Thiere, G.: Error bounds for solutions of linear equations and inequalities. Math. Methods Oper. Res. **41**, 191–214 (1995)
15. Azé, D., Corvellec, J.-N.: Characterizations of error bounds for lower semicontinuous functions on metric spaces. ESAIM Control Optim. Calc. Var. **10**, 409–425 (2004)
16. Fabian, M.J., Henrion, R., Kruger, A., Outrata, J.: Error bounds: necessary and sufficient conditions. Set Valued Anal. **18**, 121–149 (2010)
17. Kruger, A., Van Ngai, H., Théra, M.: Stability of error bounds for convex constraint systems in Banach spaces. SIAM J. Optim. **20**, 3280–3296 (2010)
18. Henrion, R., Jourani, A., Outrata, J.: On the calmness of a class of multifunctions. SIAM J. Optim. **13**, 603–618 (2002)

19. Jourani, A.: Hoffman's error bound, local controllability, and sensitivity analysis. SIAM J. Control Optim. **38**, 947–970 (2000)
20. Goberna, M.A., López, M.A.: Linear Semi-Infinite Optimization. Wiley, Chichester (1998)
21. Goberna, M.A., López, M.A.: Post-Optimal Analysis in Linear Semi-Infinite Optimization. Springer Briefs in Optimization. Springer, New York (2014)
22. Clarke, F.H.: Optimization and Nonsmooth Analysis. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. Wiley, New York (1983)
23. Luenberger, D.G., Ye, Y.: Linear and Nonlinear Programming, 3rd edn. Springer, Berlin (2008)
24. Hiriart-Urruty, J.B., Lemaréchal, C.: Convex Analysis and Minimization Algorithms I. Springer, Berlin (1991)
25. Klatte, D., Kummer, B.: On calmness of the argmin mapping in parametric optimization problems. J. Optim. Theory Appl. **165**, 708–719 (2015)