Vietnam J Math (2014) 42:467–478
DOI 10.1007/s10013-013-0052-0

A Simple Formula for the Second-Order Subdifferential of Maximum Functions

Konstantin Emich · René Henrion

Received: 30 June 2013 / Accepted: 27 September 2013 / Published online: 5 November 2013

Abstract We derive a simple formula for the second-order subdifferential of the maximum of coordinates which allows us to construct this set immediately from its argument and the direction to which it is applied. This formula can be combined with a chain rule recently proved by Mordukhovich and Rockafellar (SIAM J. Optim. 22:953–986, 2012) in order to derive a similarly simple formula for the extended partial second-order subdifferential of finite maxima of smooth functions. Analogous formulas can be derived immediately for the full and conventional partial second-order subdifferentials.

Keywords Second-order subdifferential · Extended partial second-order subdifferential · Maximum function · Calculus rules

Mathematics Subject Classification (2010) 49J52 · 49J53

1 Introduction

In 1976, B.S. Mordukhovich introduced a nonconvex normal cone and a corresponding subdifferential with the intention to derive necessary optimality conditions for optimal control problems with endpoint geometric constraints [4]. These constructions, meanwhile carrying his name, may have appeared unusual in the beginning because they fell outside the duality scheme between tangents and normals or directional derivatives and subdifferentials dominating the ideas of variational analysis at that time. Soon it became evident, however, that the renunciation of convexity was a key property for precise characterizations in dual terms of optimality conditions or stability of multifunctions. The significance of Mordukhovich's

This paper is dedicated to Prof. Boris Mordukhovich on the occasion of his 65th birthday.

K. Emich · R. Henrion (✉)
Weierstrass Institute, Mohrenstr. 39, 10117 Berlin, Germany
e-mail: henrion@wias-berlin.de

K. Emich
e-mail: emich@wias-berlin.de

normal cone and related objects such as the associated coderivative relies on the fact that they are small on the one hand but robust on the other, the latter property being the basis for a surprisingly rich calculus which made it possible to benefit from these constructions in more and more complex settings. Among the abundant proofs for the striking usefulness of the mentioned concepts, we just mention Mordukhovich's celebrated coderivative criterion for the Aubin property (and related) of general multifunctions (see, e.g., [5]). As stated in the monograph by Rockafellar/Wets [10], "the Mordukhovich criterion ... is the key to a Lipschitzian calculus for set-valued mappings". For a comprehensive account of similar results, the reader is referred to the basic monograph [6]. In the beginning, first-order variational analysis was a primary field of application. Later, the focus shifted from 'raw' objects to derived ones, e.g. from constraint mappings to solution mappings to generalized equations. A typical need for this transition arises in the derivation of dual necessary optimality conditions for hierarchical problems such as bilevel optimization or, more generally, mathematical programming with equilibrium constraints. A typical problem of interest in this context is given by

$$\min\bigl\{g(x,y) \mid y \in S(x)\bigr\}, \quad S(x) := \bigl\{y \in Y \mid 0 \in \nabla_y f(x,y) + N_C(y)\bigr\}.$$

This bilevel problem has numerous applications, for instance in the characterization of equilibria in electricity spot markets (see, e.g., [3]). Upon observing that the feasible set of this problem, i.e., the graph of S, can be equivalently described by

$$\operatorname{gr} S = \Phi^{-1}(\operatorname{gr} N_C), \quad \Phi(x,y) := \bigl(y, -\nabla_y f(x,y)\bigr),$$

it becomes evident that dual necessary optimality conditions require the analysis of normal cones to derived objects which are related to normal cones themselves. This means that one actually deals with second-order variational analysis. Corresponding indispensable calculus rules (first of all chain rules) for the so-called (full) second-order subdifferential can be found in [6]. The application of the full second-order subdifferential is limited in the setting above, however, to functions f of class $\mathcal{C}^2$ and to sets C not depending on x. Recently, however, increasing interest has arised in more general settings, e.g., when considering moving sets $C(x)$ as in the control of the sweeping process [1] or functions f being just of class $\mathcal{C}^{1,1}$ as in conditioning of linear-quadratic two-stage stochastic optimization problems [2]. In these cases it is rather the so-called *extended partial second-order subdifferential* introduced in [9] whose calculus is of interest. A fundamental chain rule for this object has been established in [9] which basically allows to reduce the computation of the extended partial second-order subdifferential of some composite function to the computation of the full second-order subdifferential of the outer function. An extensive application can be found in the recent paper [8]. An important special case of composite functions is given by maximum functions. For instance, the mapping S considered above could be generalized to

$$S(x) := \bigl\{y \in Y \mid 0 \in \partial_y f(x,y) + N_C(y)\bigr\}, \quad f(x,y) := \max_{i=1,\dots,m}\bigl\{f_i(x,y)\bigr\},$$

where the f_i are possibly smooth. A potential application would again arise from electricity spot market models with convex, nonsmooth bidding functions. Then, f is no longer differentiable and so one deals with multi-valued base mappings. Lipschitz properties of mappings S with such multi-valued base have been studied in [7].

In the context of such maximum functions, the chain rule mentioned above requires an efficient computation of the full second-order subdifferential of the maximum of coordinates. In principle, this can be realized by using a formula provided in [9, Lemma 4.4]. A concrete application of this formula, however, requires to evaluate differences of certain critical faces to the standard simplex (see (4)), which may be tedious work within a systematic study. In this paper we present a very simple formula for the second-order subdifferential of the maximum of coordinates which provides an immediate construction given the initial data, i.e., the point in the graph and the direction of interest. The obtained formula is then used to prove a similarly immediate expression for the extended partial second-order subdifferential of a finite maximum of smooth functions.

2 Basic Concepts and Notation

As usual, we denote by 'gr M' the graph of some multifunction M, by $\overline{\mathbb{R}}$ the extended real line, i.e., $\overline{\mathbb{R}} := [-\infty, +\infty]$, and by Z^o the polar cone of some set Z, i.e. $Z^o := \{y \mid \langle y, z\rangle \leq 0 \ \forall z \in Z\}$. We recall the following definitions [6].

Definition 2.1 Let $C \subseteq \mathbb{R}^m$ be a closed subset and $\bar{x} \in C$. The Mordukhovich normal cone to C at $\bar{x}$ is defined by

$$N_C(\bar{x}) := \{x^* \mid \exists (x_n, x_n^*) \to (\bar{x}, x^*) : x_n \in C, x_n^* \in [T_C(x_n)]^o\}.$$

Here, $[T_C(x)]^o$ refers to the Fréchet normal cone to C at x, which is the polar of the contingent cone

$$T_C(x) := \{d \in \mathbb{R}^m \mid \exists t_k \downarrow 0, d_k \to d : x + t_k d_k \in C \ \forall k\}$$

to C at x. For an extended-real-valued, lower semicontinuous function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ with $|f(\bar{x})| < \infty$, the Mordukhovich normal cone induces a subdifferential via

$$\partial f(\bar{x}) := \{x^* \mid (x^*, -1) \in N_{\mathrm{epi} f}(\bar{x}, f(\bar{x}))\}.$$

Definition 2.2 Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction with closed graph. The Mordukhovich coderivative $D^* M(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of M at some $(\bar{x}, \bar{y}) \in \mathrm{gr} M$ is defined as

$$D^* M(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n \mid (x^*, -y^*) \in N_{\mathrm{gr} M}(\bar{x}, \bar{y})\}.$$

Definition 2.3 For a lower semicontinuous function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ which is finite at $x \in \mathbb{R}^n$ and for an element $s \in \partial f(x)$ the second-order subdifferential of f at x relative to s is a multifunction $\partial^2 f(x, s) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial^2 f(x, s)(u) := (D^* \partial f)(x, s)(u) \quad \forall u \in \mathbb{R}^n.$$

Definition 2.4 For a lower semicontinuous function $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ which is finite at $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, the partial subdifferential is defined as $\partial_y f(x, y) := \partial f(x, \cdot)(y)$. Following [9], for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and any $s \in \partial_y f(x, y)$ the (extended) partial second-order subdifferential of f with respect to y at (x, y) relative to s is a multifunction $\tilde{\partial}_y^2 f(x, y, s) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$\tilde{\partial}_y^2 f(x, y, s)(u) := (D^* \partial_y f)(x, y, s)(u) \quad \forall u \in \mathbb{R}^m.$$

3 A Formula for the Second-Order Subdifferential of the Maximum of Coordinates

Let $\theta : \mathbb{R}^m \to \mathbb{R}$ be defined as

$$\theta(x) := \max_{i=1,\dots,m} x_i.$$

Our aim is to calculate the second-order subdifferential of θ. We denote by $J(x)$ the set of active indices at x:

$$J(x) := \big\{ i \in \{1,\dots,m\} \mid x_i = \theta(x) \big\}$$

and by $J^c(x)$ its complement. Furthermore for any fixed $(\bar{x},\bar{s}) \in \operatorname{gr} \partial\theta$ we introduce the following index sets

$$K := \big\{ i \in J(\bar{x}) \mid \bar{s}_i = 0 \big\}, \qquad L := \big\{ i \in J(\bar{x}) \mid \bar{s}_i > 0 \big\}.$$

Theorem 3.1 *For any fixed $(\bar{x},\bar{s}) \in \operatorname{gr} \partial\theta$ and any fixed $u \in \mathbb{R}^m$ the second-order subdifferential $\partial^2\theta(\bar{x},\bar{s})(u)$ is nonempty if and only if $u_i = c$ for all $i \in L$, where c is a constant. In this case we have*

$$\partial^2\theta(\bar{x},\bar{s})(u) = \left\{ w \;\middle|\; \sum_{i=1}^m w_i = 0,\, w_i \geq 0\ \forall i \in J_>,\, w_i = 0\ \forall i \in J^c(\bar{x}) \cup J_< \right\}, \tag{1}$$

where $J_> := \{i \in J(\bar{x}) \mid u_i > c\}$ and $J_< := \{i \in J(\bar{x}) \mid u_i < c\}$.

Proof We first notice that the function θ can be written in the equivalent form

$$\theta(x) = \max_{v \in S} \langle v, x \rangle, \quad \text{where } S := \left\{ s \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^m s_i = 1,\, s_i \geq 0,\, i = 1,\dots,m \right\}. \tag{2}$$

Denoting by e_i the canonical unit vectors of $\mathbb{R}^m$, the subdifferential of θ at $\bar{x}$ is well known to admit the representation:

$$\partial\theta(\bar{x}) = \operatorname{conv}\big\{ e_i \mid i \in J(\bar{x}) \big\} \subseteq S.$$

Accordingly, the fixed $(\bar{x},\bar{s}) \in \operatorname{gr} \partial\theta$ satisfies

$$\sum_{i=1}^m \bar{s}_i = 1, \quad \bar{s}_i > 0\ \forall i \in L,\ \bar{s}_i = 0\ \forall i \in J^c(\bar{x}) \cup K. \tag{3}$$

In particular, $L \neq \emptyset$. It is easy to see that the contingent cone $T_S(\bar{s})$ to the simplex S at $\bar{s} \in \partial\theta(\bar{x})$ is given by

$$T_S(\bar{s}) = \left\{ h \;\middle|\; \sum_{i=1}^m h_i = 0,\, h_i \geq 0\ \forall i \in J^c(\bar{x}) \cup K \right\}.$$

By [9, Lemma 4.4], the second-order subdifferential of θ at $(\bar{x},\bar{s})$ in direction u can be characterized as follows:

$$\begin{aligned} \partial^2\theta(\bar{x},\bar{s})(u) = \big\{ w \mid & \text{there are closed faces } K_1 \supseteq K_2 \text{ of } T_S(\bar{s}) \cap \bar{x}^\perp \text{ such that} \\ & w \in K_1 - K_2,\, -u \in (K_1 - K_2)^o \big\}. \end{aligned} \tag{4}$$

Recall that a *closed face* K of a polyhedral cone C is a polyhedral cone of the form

$$K := \{x \in C \mid \langle x, c\rangle = 0\} \quad \text{for some } c \in C^o. \tag{5}$$

We claim that the *critical cone* to S at $\bar{s}$ is given by

$$T_S(\bar{s}) \cap \bar{x}^\perp = \left\{ h \,\middle|\, \sum_{i=1}^m h_i = 0, h_i \geq 0\ \forall i \in K, h_i = 0\ \forall i \in J^c(\bar{x}) \right\}. \tag{6}$$

Indeed, if $h \in T_S(\bar{s}) \cap \bar{x}^\perp$, then, by $h_i \geq 0$ for all $i \in J^c(\bar{x})$,

$$0 = \langle \bar{x}, h\rangle = \theta(\bar{x}) \sum_{i \in J(\bar{x})} h_i + \sum_{i \in J^c(\bar{x})} \bar{x}_i h_i \leq \theta(\bar{x}) \sum_{i=1}^m h_i = 0 \tag{7}$$

which implies that the inequality above actually holds as an equality and, thus,

$$\sum_{i \in J^c(\bar{x})} h_i \big(\bar{x}_i - \theta(\bar{x})\big) = 0.$$

Since $\bar{x}_i < \theta(\bar{x})$ for $i \in J^c(\bar{x})$, it follows that $h_i = 0$ for all $i \in J^c(\bar{x})$, whence h belongs to the right-hand side of (6). Conversely, if h belongs to the right-hand side of (6), then clearly $h \in T_S(\bar{s})$. Now, (7) can be read from the right to the left in order to derive that $h \in \bar{x}^\perp$. This proves (6).

The application of (4) requires the knowledge of the closed faces of $T_S(\bar{s}) \cap \bar{x}^\perp$. From the general definition of a closed face in (5) it follows that the closed faces of $T_S(\bar{s}) \cap \bar{x}^\perp$ obtained by turning subsets of inequalities in (6) into equalities. Accordingly, the closed faces of $T_S(\bar{s}) \cap \bar{x}^\perp$ are given by

$$M_I := \left\{ h \,\middle|\, \sum_{i=1}^m h_i = 0, h_i \geq 0\ \forall i \in K \setminus I, h_i = 0\ \forall i \in J^c(\bar{x}) \cup I \right\} \quad (I \subseteq K).$$

We show that

$$I_1 \subseteq I_2 \quad \Leftrightarrow \quad M_{I_1} \supseteq M_{I_2} \quad \forall I_1, I_2 \subseteq K. \tag{8}$$

The direction '$\Rightarrow$' is evident. For the reverse direction let $I_1, I_2 \subseteq K$ and assume that $I_1 \nsubseteq I_2$. Then there exists some $i' \in I_1 \setminus I_2$. As observed below (3), we may choose some index $i^* \in L$. Recall that $K \cap L = \emptyset$. Hence, $i^* \notin K$ and $i' \neq i^*$. This allows us to define h by setting $h_{i'} := 1, h_{i^*} := -1$ and $h_i := 0$ for all remaining i. Then, $h_i \geq 0$ for all $i \in K \setminus I_2$ due to $i^* \notin K$. Furthermore, $i', i^* \notin I_2$ and $i', i^* \in J(\bar{x})$ due to $K, L \subseteq J(\bar{x})$. This implies that $h_i = 0$ for all $i \in J^c(\bar{x}) \cup I_2$, whence $h \in M_{I_2}$. On the other hand, $i' \in I_1$, so that $h_{i'} = 1$ leads to $h \notin M_{I_1}$. Altogether this proves (8).

Combining (8) with (4), we arrive at

$$\partial^2\theta(\bar{x}, \bar{s})(u) = \{w \mid \exists I_1 \subseteq I_2 \subseteq K : w \in M_{I_1} - M_{I_2}, -u \in (M_{I_1} - M_{I_2})^o\}. \tag{9}$$

We show next that for $I_1 \subseteq I_2 \subseteq K$ the sets $M_{I_1} - M_{I_2}$ and $(M_{I_1} - M_{I_2})^o$ are given by

$$M_{I_1} - M_{I_2} = \left\{ p \,\middle|\, \sum_{i=1}^m p_i = 0, p_i \geq 0\ \forall i \in I_2 \setminus I_1, p_i = 0\ \forall i \in J^c(\bar{x}) \cup I_1 \right\} \tag{10}$$

and

$$(M_{I_1} - M_{I_2})^o = \left\{p^* \mid \exists \tilde{c} \in \mathbb{R} : p_i^* \leq \tilde{c} \; \forall i \in I_2 \setminus I_1, \, p_i^* = \tilde{c} \; \forall i \in J(\bar{x}) \setminus I_2\right\}. \tag{11}$$

The inclusion '$\subseteq$' in (10) follows readily from the definitions. For the reverse inclusion, let p belong to the right-hand side of (10)) and define $h^{(1)}, h^{(2)}$ by

$$\begin{aligned}
&h_i^{(1)} := p_i, \quad h_i^{(2)} := 0 \quad \forall i \in J^c(\bar{x}) \cup I_2, \\
&h_i^{(1)} := [p_i]_+, \quad h_i^{(2)} := [p_i]_- \quad \forall i \in J(\bar{x}) \backslash \left(I_2 \cup \left\{i^*\right\}\right), \\
&h_{i^*}^{(1)} := -\sum_{i \in J^c(\bar{x}) \cup I_2} p_i - \sum_{i \in J(\bar{x}) \backslash (I_2 \cup \{i^*\})} [p_i]_+, \quad h_{i^*}^{(2)} := -\sum_{i \in J(\bar{x}) \backslash (I_2 \cup \{i^*\})} [p_i]_-,
\end{aligned}$$

where $[p_i]_+$, $[p_i]_-$ denote the positive and negative parts of p_i, respectively, and i^* is an arbitrarily selected index of the nonempty set $L \subseteq J(\bar{x})$ as it has already been used before. Then, by construction, $\sum_{i=1}^m h_i^{(1)} = 0$. The properties of p and the inclusion $I_1 \subseteq I_2$ ensure that $h_i^{(1)} = p_i = 0$ for all $i \in J^c(\bar{x}) \cup I_1$. Finally, if $i \in K \backslash I_1$ is arbitrary then $i \neq i^*$ due to $K \cap L = \emptyset$. We distinguish the cases $i \notin I_2$ and $i \in I_2$. In the first case it follows that $i \in J(\bar{x}) \backslash (I_2 \cup \{i^*\})$ by $K \subseteq J(\bar{x})$. Then, $h_i^{(1)} := [p_i]_+ \geq 0$. Otherwise, $i \in I_2 \backslash I_1$ and $h_i^{(1)} = p_i \geq 0$. Summarizing, $h_i^{(1)} \geq 0$ for all $i \in K \backslash I_1$ and, thus, $h^{(1)} \in M_{I_1}$. Similarly, by construction, $\sum_{i=1}^m h_i^{(2)} = 0$ and $h_i^{(2)} = 0$ for all $i \in J^c(\bar{x}) \cup I_2$. Moreover, if $i \in K \backslash I_2$ is arbitrary then $i \neq i^*$ due to $K \cap L = \emptyset$. Therefore, $i \in J(\bar{x}) \backslash (I_2 \cup \{i^*\})$ by $K \subseteq J(\bar{x})$. We conclude that $h_i^{(2)} = [p_i]_- \geq 0$, whence $h^{(2)} \in M_{I_2}$. It remains to show that $p = h^{(1)} - h^{(2)}$. Indeed, by construction one has the following exhaustive cases:

$$\begin{aligned}
&h_i^{(1)} - h_i^{(2)} = p_i \quad \forall i \in J^c(\bar{x}) \cup I_2, \\
&h_i^{(1)} - h_i^{(2)} = [p_i]_+ - [p_i]_- = p_i \quad \forall i \in J(\bar{x}) \backslash \left(I_2 \cup \left\{i^*\right\}\right), \\
&h_{i^*}^{(1)} - h_{i^*}^{(2)} = -\sum_{i \in J^c(\bar{x}) \cup I_2} p_i - \sum_{i \in J(\bar{x}) \backslash (I_2 \cup \{i^*\})} p_i = p_{i^*}.
\end{aligned}$$

Here, the last equality follows from $\sum_{i=1}^m p_i = 0$. Altogether this proves (10).

By duality theory for systems of linear equalities and inequalities we have $p^* \in (M_{I_1} - M_{I_2})^o$ if and only if there are coefficients $\tilde{c}, \lambda_i \in \mathbb{R}$ for $i \in J^c(\bar{x}) \cup I_1$ and $\mu_i \leq 0$ for $i \in I_2 \setminus I_1$ such that

$$p^* = \tilde{c}\mathbf{1} + \sum_{i \in J^c(\bar{x}) \cup I_1} \lambda_i e_i + \sum_{i \in I_2 \backslash I_1} \mu_i e_i,$$

where $\mathbf{1} = (1, \ldots, 1)^T$. For $j \in I_2 \setminus I_1$ it follows that $j \notin J^c(\bar{x}) \cup I_1$, whence $p_j^* = \tilde{c} + \mu_j \leq \tilde{c}$. For $j \in J(\bar{x}) \setminus I_2$ it follows that $j \notin J^c(\bar{x}) \cup I_1$ (due to $I_1 \subseteq I_2$) and also $j \notin I_2 \setminus I_1$, whence $p_j^* = \tilde{c}$. Summarizing, $p^* \in (M_{I_1} - M_{I_2})^o$ if and only if it satisfies the conditions on the right-hand side of (11).

$$(M_{I_1} - M_{I_2})^o = \left\{p^* \;\middle|\; J(\bar{x}) \setminus I_2 \subseteq \left\{i \in J(\bar{x}) \;\middle|\; p_i^* = \max_{j \in J(\bar{x}) \backslash I_1} p_j^*\right\}\right\}.$$

By (9) we have

$$\partial^2\theta(\bar{x},\bar{s})(u) = \Bigg\{ w \,\Bigg|\, \exists I_1 \subseteq I_2 \subseteq K : \sum_{i=1}^m w_i = 0,\, w_i \geq 0\ \forall i \in I_2 \setminus I_1,$$

$$w_i = 0\ \forall i \in J^c(\bar{x}) \cup I_1,\, J(\bar{x}) \setminus I_2 \subseteq \Big\{ i \in J(\bar{x}) \,\Big|\, u_i = \min_{j \in J(\bar{x})\setminus I_1} u_j \Big\} \Bigg\}. \quad (12)$$

With the notations

$$A_{I_1,I_2} := \Bigg\{ w \,\Bigg|\, \sum_{i=1}^m w_i = 0,\, w_i \geq 0\ \forall i \in I_2 \setminus I_1,\, w_i = 0\ \forall i \in J^c(\bar{x}) \cup I_1 \Bigg\}, \quad (13)$$

$$J(I) = \Big\{ i \in J(\bar{x}) \,\Big|\, u_i = \min_{j \in J(\bar{x})\setminus I} u_j \Big\} \quad (14)$$

(12) can be written as

$$\partial^2\theta(\bar{x},\bar{s})(u) = \bigcup_{\substack{I_1\subseteq I_2\subseteq K \\ J(\bar{x})\setminus I_2 \subseteq J(I_1)}} A_{I_1,I_2} = \bigcup_{\substack{I_1\subseteq I_2\subseteq K \\ J(\bar{x})\setminus J(I_1) \subseteq I_2}} A_{I_1,I_2} = \bigcup_{\substack{I_1\subseteq K \\ J(\bar{x})\setminus J(I_1) \subseteq K}} \Bigg\{ \bigcup_{\substack{I_1\subseteq I_2\subseteq K \\ J(\bar{x})\setminus J(I_1) \subseteq I_2}} A_{I_1,I_2} \Bigg\}.$$

In the iterated union on the right-hand side we consider the inner union over sets I_2 to be conditional with respect to the set $I_1 \subseteq K$ fixed in the outer union. Note that in case of $J(\bar{x}) \setminus J(I_1) \not\subseteq K$ there exists no I_2 satisfying the conditions of the inner union which allows us to add the condition $J(\bar{x}) \setminus J(I_1) \subseteq K$ to the outer union. Now, for any fixed I_1 satisfying the conditions of the outer union let $I_2^a \subseteq I_2^b$ be two index sets satisfying the conditions for I_2 in the inner union. By (13), it then holds that $A_{I_1,I_2^a} \supseteq A_{I_1,I_2^b}$. Thus, one may shrink the inner union to the minimal set I_2 of the indicated family which is evidently given by $I_2 = I_1 \cup (J(\bar{x}) \setminus J(I_1))$. Consequently, for $M := \{I \subseteq K \mid J(\bar{x}) \setminus J(I) \subseteq K\}$ we have

$$\partial^2\theta(\bar{x},\bar{s})(u) = \bigcup_{I\in M} A_{I,I\cup(J(\bar{x})\setminus J(I))} = \bigcup_{I\in M} \tilde{A}_I \quad \text{with } \tilde{A}_I := A_{I,I\cup(J(\bar{x})\setminus J(I))} \quad (I \in M). \quad (15)$$

From $J(\bar{x}) \setminus K = L$ and (14) we derive that

$$M = \big\{ I \subseteq K \mid L \subseteq J(I) \big\} = \Big\{ I \subseteq K \mid u_i = \min_{j \in J(\bar{x})\setminus I} u_j\ \forall i \in L \Big\}. \quad (16)$$

Note that $J(\bar{x}) \setminus I \neq \emptyset$ for all $I \in M$ because of $I \subseteq K$ and, thus, $\emptyset \neq L = J(\bar{x}) \setminus K \subseteq J(\bar{x}) \setminus I$. This implies that M is nonempty if and only if u_i is constant for all $i \in L$ which is the first assertion of the theorem.

To show the asserted formula (1), assume now that $M \neq \emptyset$, hence $u_i = c$ for all $i \in L$ and for some constant c. Consequently, for $J_<$ as introduced below (1), one has $J_< \subseteq J(\bar{x}) \setminus L = K$. Moreover, by (16),

$$I \in M \iff I \subseteq K \text{ and } u_j \geq c\ \forall j \in J(\bar{x}) \setminus I \iff J_< \subseteq I \subseteq K. \quad (17)$$

In particular, because of $J_< \subseteq K$, we infer that $J_< \in M$ and that $J_<$ is in fact the smallest member of M.

In the last step of the proof we show that $\tilde{A}_I$ defined in (15) is a decreasing family of sets. To this aim, we define $J_I := J(\bar{x}) \setminus (J(I) \cup I)$ for $I \in M$, and observe that $J_I =$

$[I \cup (J(\bar{x}) \setminus J(I))] \setminus I$, whence by definition of $\tilde{A}_I$

$$\tilde{A}_I = \left\{ w \;\middle|\; \sum_{i=1}^{m} w_i = 0,\, w_i \geq 0\; \forall i \in J_I,\, w_i = 0\; \forall i \in J^c(\bar{x}) \cup I \right\}. \tag{18}$$

From the definitions of J_I and $J(I)$ we conclude that

$$J_I = \left\{ i \in J(\bar{x}) \setminus I \mid u_i > \min_{j \in J(\bar{x}) \setminus I} u_j \right\}. \tag{19}$$

Now let $I^a, I^b \in M$ be arbitrary with $I^a \subseteq I^b$. We claim that $\tilde{A}_{I^b} \subseteq \tilde{A}_{I^a}$. For arbitrarily given $w \in \tilde{A}_{I^b}$ one has by (18) that

$$\sum_{i=1}^{m} w_i = 0, \quad w_i \geq 0\; \forall i \in J_{I^b},\, w_i = 0\; \forall i \in J^c(\bar{x}) \cup I^b. \tag{20}$$

Since $I^a \subseteq I^b$, it is sufficient to show that $w_i \geq 0$ for all $i \in J_{I^a} \setminus J_{I^b}$. Let such i be arbitrarily given. Then, by (19), $i \in J(\bar{x}) \setminus I^a$ and by (16)

$$u_i > \min_{j \in J(\bar{x}) \setminus I^a} u_j = c. \tag{21}$$

Moreover, $i \notin J_{I^b}$ leaves two possibilities according to (19): either $i \notin J(\bar{x}) \setminus I^b$ which by $i \in J(\bar{x}) \setminus I^a$ leads to $i \in I^b \setminus I^a$, whence $w_i = 0$ (see (20)); or, again relying on (16),

$$u_i \leq \min_{j \in J(\bar{x}) \setminus I^b} u_j = c,$$

which is a contradiction with (21). Summarizing, $w_i \geq 0$ and, thus, $\tilde{A}_{I^b} \subseteq \tilde{A}_{I^a}$.

The fact that $\tilde{A}_I$ defined in (15) is a decreasing family of sets yields along with (17) that

$$\begin{aligned} \partial^2\theta(\bar{x}, \bar{s})(u) &= \bigcup_{I \in M} \tilde{A}_I = \tilde{A}_{J_<} \\ &= \left\{ w \;\middle|\; \sum_{i=1}^{m} w_i = 0,\, w_i \geq 0\; \forall i \in J_{J_<},\, w_i = 0\; \forall i \in J^c(\bar{x}) \cup J_< \right\} \end{aligned}$$

by (18). Observing that by (19), (17) and (16)

$$J_{J_<} = \left\{ i \in J(\bar{x}) \setminus J_< \mid u_i > c \right\} = \left\{ i \in J(\bar{x}) \mid u_i > c \right\} = J_>$$

for $J_>$ being defined below (1), we finally derive the asserted formula (1). □

Corollary 3.1 *In the setting of Theorem 3.1 one has*

$$\partial^2\theta(\bar{x}, \bar{s})(0) = \left\{ w \;\middle|\; \sum_{i=1}^{m} w_i = 0,\, w_i = 0\; \forall i \in J^c(\bar{x}) \right\}.$$

Proof Since $L \neq \emptyset$, it follows that $c = 0$ for the constant c defined in the statement of Theorem 3.1. Accordingly, $J_< = J_> = \emptyset$ and the assertion follows from (1). □

In order to illustrate how Theorem 3.1 reduces the effort to calculate the second-order subdifferential of a maximum of coordinates, we pick up an example from [9]:

Example 3.1 ([9, Example 3.5]) Let $m = 4$, $\bar{x} = (0, 0, 0, 0)$, $\bar{s} = (\frac{1}{2}, \frac{1}{2}, 0, 0)$, $u = (0, 0, 1, -1)$. Then, $J(\bar{x}) = \{1, 2, 3, 4\}$, $J^c(\bar{x}) = \{\emptyset\}$ and $L = \{1, 2\}$. Since $u_i = 0 =: c$ for all $i \in L$, it follows that $J_> = \{3\}$, $J_< = \{4\}$. Thus,

$$\partial^2\theta(\bar{x}, \bar{s})(u) = \left\{ w \,\middle|\, \sum_{i=1}^{4} w_i = 0, w_3 \geq 0, w_4 = 0 \right\}.$$

4 A Formula for the Extended Partial Second-Order Subdifferential of a Finite Maximum of Smooth Functions

We are now going to apply the result of Theorem 3.1 to the maximum not just of coordinates but of general smooth functions:

$$\varphi(x, y) := \max_{i=1,\dots,m} g_i(x, y).$$

We partition the argument into two parts in order to discuss the extended second-order subdifferential of φ. A completely analogous result will hold true for the conventional second-order subdifferential. Clearly we may write $\varphi = \theta \circ g$ with $\theta(z_1, \dots, z_m) := \max_{i=1,\dots,m} z_i$ and $g(x, y) := (g_1(x, y), \dots, g_m(x, y))^T$. We assume that $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^m$ is of class $\mathcal{C}^2$. The key tool for our result is the following chain rule:

Theorem 4.1 ([9, Theorems 3.3 & 4.3]) *Fix any* $(\bar{x}, \bar{y})$ *and* $\bar{s} \in \partial_y\varphi(\bar{x}, \bar{y})$. *Assume that the basic second-order constraint qualification*

$$\partial^2\theta\big(g(\bar{x}, \bar{y}), v\big)(0) \cap \ker \nabla_y^T g(\bar{x}, \bar{y}) = \{0\} \tag{22}$$

is satisfied at some $v \in \partial\theta(g(\bar{x}, \bar{y}))$ *with* $\bar{s} = \nabla_y^T g(\bar{x}, \bar{y})v$. *Then,* v *is uniquely defined and for all* $u \in \mathbb{R}^d$ *we have*

$$\tilde{\partial}_y^2\varphi(\bar{x}, \bar{y}, \bar{s})(u) = \begin{bmatrix} \nabla_{xy}^2\langle v, g\rangle(\bar{x}, \bar{y}) \\ \nabla_{yy}^2\langle v, g\rangle(\bar{x}, \bar{y}) \end{bmatrix} u + \begin{bmatrix} \nabla_x^T g(\bar{x}, \bar{y}) \\ \nabla_y^T g(\bar{x}, \bar{y}) \end{bmatrix} \partial^2\theta\big(g(\bar{x}, \bar{y}), v\big)\big(\nabla_y g(\bar{x}, \bar{y})u\big). \tag{23}$$

The basic second-order constraint qualification (22) motivates us to consider the following well-known weakened form of linear independence:

Definition 4.1 A set $\{z_1, \dots, z_s\}$ of vectors is called *affinely independent* if the set of enhanced vectors $\{(z_1, 1), \dots, (z_s, 1)\}$ is linearly independent in the classical sense.

Assume that the set of active gradients $\{\nabla_y g_i(\bar{x}, \bar{y})\}_{i \in I(\bar{x}, \bar{y})}$ is affinely independent. Recalling that

$$\tilde{\partial}_y\varphi(\bar{x}, \bar{y}) = \operatorname{conv}\big\{\nabla_y g_i(\bar{x}, \bar{y})\big\}_{i \in I(\bar{x}, \bar{y})}, \quad \text{where } I(\bar{x}, \bar{y}) := \left\{ i \,\middle|\, g_i(\bar{x}, \bar{y}) = \max_j g_j(\bar{x}, \bar{y}) \right\},$$

we observe that any $\bar{s} \in \tilde{\partial}_y\varphi(\bar{x}, \bar{y})$ can be written in the form

$$\bar{s} = \sum_{i \in I(\bar{x}, \bar{y})} v_i \nabla_y g_i(\bar{x}, \bar{y}), \qquad \sum_{i \in I(\bar{x}, \bar{y})} v_i = 1$$

or

$$(\bar{s}, 1) = \sum_{i \in I(\bar{x},\bar{y})} v_i \big(\nabla_y g_i(\bar{x}, \bar{y}), 1\big).$$

It follows that the multipliers v_i are uniquely defined by $\bar{s}$. Now, we are in a position to make formula (23) for the extended second-order partial subdifferential fully explicit:

Theorem 4.2 *In the setting of Theorem 4.1 assume that the set* $\{\nabla_y g_i(\bar{x}, \bar{y})\}_{i \in I(\bar{x},\bar{y})}$ *is affinely independent. Denote by* $v \in \partial\theta(g(\bar{x}, \bar{y}))$ *the unique element defined by* $\bar{s} = \nabla_y^T g(\bar{x}, \bar{y})v$ *(see discussion above) and let* $L := \{i \in I(\bar{x}, \bar{y}) \mid v_i > 0\}$. *Then, for arbitrary* $u \in \mathbb{R}^d$, *one has*

$$\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) \neq \emptyset \iff \exists c \in \mathbb{R} : \big\langle \nabla_y g_i(\bar{x}, \bar{y}), u\big\rangle = c \quad \forall i \in L. \tag{24}$$

In this case we have

$$\tilde{w} \in \tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) \iff \begin{cases} \exists w \in \mathbb{R}^m : \\ \tilde{w} = \begin{bmatrix} \nabla_{xy}^2 \langle v, g\rangle(\bar{x}, \bar{y}) \\ \nabla_{yy}^2 \langle v, g\rangle(\bar{x}, \bar{y}) \end{bmatrix} u + \begin{bmatrix} \nabla_x^T g(\bar{x}, \bar{y}) \\ \nabla_y^T g(\bar{x}, \bar{y}) \end{bmatrix} w, \\ \sum_{i=1}^m w_i = 0, \\ w_i \geq 0 \quad \forall i \in I_>, \\ w_i = 0 \quad \forall i \in I^c(\bar{x}, \bar{y}) \cup I_<, \end{cases} \tag{25}$$

where

$$\begin{aligned} I_> &:= \big\{i \in I(\bar{x}, \bar{y}) | \big\langle \nabla_y g_i(\bar{x}, \bar{y}), u\big\rangle > c\big\}, \\ I_< &:= \big\{i \in I(\bar{x}, \bar{y}) | \big\langle \nabla_y g_i(\bar{x}, \bar{y}), u\big\rangle < c\big\}, \\ I^c(\bar{x}, \bar{y}) &:= \{1, \ldots, m\} \setminus I(\bar{x}, \bar{y}). \end{aligned}$$

Proof By Corollary 3.1 we have

$$\partial^2\theta\big(g(\bar{x}, \bar{y}), v\big)(0) = \left\{ w \,\middle|\, \sum_{i=1}^m w_i = 0, w_i = 0 \; \forall i \in I^c(\bar{x}, \bar{y}) \right\}. \tag{26}$$

Then, evidently the inclusion '$\supseteq$' in (22) is satisfied. Conversely, if w belongs to the left-hand side of (22), then w satisfies (26) and $w \in \ker \nabla_y^T g(\bar{x}, \bar{y})$. Hence,

$$\sum_{i \in I(\bar{x},\bar{y})} w_i \big(\nabla_y g_i(\bar{x}, \bar{y}), 1\big) = \sum_{i=1}^m w_i \big(\nabla_y g_i(\bar{x}, \bar{y}), 1\big) = (0, 0).$$

The assumed affine independence of the set $\{\nabla_y g_i(\bar{x}, \bar{y})\}_{i \in I(\bar{x},\bar{y})}$ provides that $w_i = 0$ for all $i \in I(\bar{x}, \bar{y})$. Along with $w_i = 0$ for all $i \in I^c(\bar{x}, \bar{y})$ by (26), we arrive at the desired conclusion $w = 0$. Thus, (22) is satisfied and we may invoke Theorem 4.1 in order to apply (23). Our first conclusion from (23) is that

$$\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) \neq \emptyset \iff \partial^2\theta\big(g(\bar{x}, \bar{y}), v\big)\big(\nabla_y g(\bar{x}, \bar{y})u\big) \neq \emptyset.$$

By virtue of Theorem 3.1 this entails (24). Similarly, (25) follows from (23) upon using the explicit representation (1). □

Remark 4.1 Exploiting similar chain rules to the one given in Theorem 4.1 but relating to the full and conventional partial second-order subdifferential rather than to its extended partial one (see [6, 9]) one may derive in the same way corresponding fully explicit formulas for those other types of second-order subdifferentials of finite maxima of smooth functions.

As an illustration of Theorem 4.2 we consider the following example:

Example 4.1 Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) := (-xy, xy^2)$ and consider the point $(\bar{x}, \bar{y}) := (1, 0)$. Then, $\varphi(\bar{x}, y) = \max\{-y, y^2\}$ and $\partial_y \varphi(\bar{x}, \bar{y}) = [-1, 0]$. Moreover, $I(\bar{x}, \bar{y}) = \{1, 2\}$. Due to $\nabla_y g_1(\bar{x}, \bar{y}) = -1$ and $\nabla_y g_2(\bar{x}, \bar{y}) = 0$ the active gradients are affinely independent (while not linearly independent) as required in Theorem 4.2. Clearly, for any $\bar{s} \in \partial_y \varphi(\bar{x}, \bar{y})$ the vector v has the unique representation $v = (-\bar{s}, 1 + \bar{s})$. Now, (24) yields $\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) \neq \emptyset$ if $L = \{1\}$ or $L = \{2\}$ or finally, if $L = \{1, 2\}$ and $u = 0$. In these cases, formula (25) reduces with our concrete data to

$$\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) = \left\{ \begin{pmatrix} -v_1 u \\ 2v_2 u - w_1 \end{pmatrix} \middle| \, w_1 = -w_2,\, w_i \geq 0 (i \in I_>),\, w_i = 0 \ (i \in I_<) \right\}.$$

Now, let $u \in \mathbb{R}$ be arbitrary. We consider three cases:

1. $\bar{s} = -1$. Then, $v = (1, 0), L = \{1\}, c = -u$. If $u > 0$, then $I_> = \{2\}$ else $I_> = \emptyset$. Similarly, if $u < 0$, then $I_< = \{2\}$ else $I_< = \emptyset$. Thus,

$$\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) = \{(-u, t) \mid t \in A\}, \quad \text{where } A = \begin{cases} t \geq 0 & \text{if } u > 0, \\ t \in \mathbb{R} & \text{if } u = 0, \\ t = 0 & \text{if } u < 0. \end{cases}$$

2. $\bar{s} = 0$. Then, $v = (0, 1), L = \{2\}, c = 0$. If $u < 0$, then $I_> = \{1\}$ else $I_> = \emptyset$. Similarly, if $u > 0$, then $I_< = \{1\}$ else $I_< = \emptyset$. Thus,

$$\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) = \{(0, 2u - t) \mid t \in A\}, \quad \text{where } A = \begin{cases} t = 0 & \text{if } u > 0, \\ t \in \mathbb{R} & \text{if } u = 0, \\ t \geq 0 & \text{if } u < 0. \end{cases}$$

3. $-1 < \bar{s} < 0$. Then, $v = (-\bar{s}, 1 + \bar{s}), L = \{1, 2\}, c = -u$. As mentioned above, $\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) = \emptyset$ if $u \neq 0$. If $u = 0$ then $I_> = I_< = \emptyset$. Thus, $\tilde{\partial}_y^2 \varphi(\bar{x}, \bar{y}, \bar{s})(u) = \{0\} \times \mathbb{R}$.

Acknowledgements This work was supported by the DFG Research Center MATHEON “Mathematics for key technologies” in Berlin.

References

1. Colombo, G., Henrion, R., Hoang, N.D., Mordukhovich, B.S.: Optimal control of the sweeping process. Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms **19**, 117–159 (2012)
2. Emich, K., Henrion, R., Römisch, W.: Conditioning of linear-quadratic two-stage stochastic optimization problems. Weierstrass Institute, Preprint No. 1783 (2013)
3. Henrion, R., Outrata, J., Surowiec, T.: On regular coderivatives in parametric equilibria with non-unique multiplier. Math. Program. **136**, 111–131 (2012)

4. Mordukhovich, B.S.: Maximum principle in problems of time optimal control with nonsmooth constraints. J. Appl. Math. Mech. **40**, 960–969 (1976)
5. Mordukhovich, B.S.: Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions. Trans. Am. Math. Soc. **340**, 1–36 (1993)
6. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation I: Basic Theory. Grundlehren der mathematischen Wissenschaften, vol. 330. Springer, Berlin (2006)
7. Mordukhovich, B.S., Nam, N.M.: Variational analysis of extended generalized equations via coderivative calculus in Asplund spaces. J. Math. Anal. Appl. **350**, 663–679 (2009)
8. Mordukhovich, B.S., Nam, N.M., Nhi, N.T.Y.: Partial second-order subdifferentials in variational analysis and optimization. Numer. Funct. Anal. Optim. (to appear)
9. Mordukhovich, B.S., Rockafellar, R.T.: Second-order subdifferential calculus with applications to tilt stability in optimization. SIAM J. Optim. **22**, 953–986 (2012)
10. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Grundlehren der mathematischen Wissenschaften, vol. 317. Springer, Berlin (1998)