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Lipschitz lower semicontinuity moduli for linear inequality systems ☆

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ABSTRACT

The paper is focussed on the Lipschitz lower semicontinuity of the feasible set mapping for linear (finite and infinite) inequality systems in three different perturbation frameworks: full, right-hand side and left-hand side perturbations. Inspired by [14], we introduce the Lipschitz lower semicontinuity-star as an intermediate notion between the Lipschitz lower semicontinuity and the well-known Aubin property. We provide explicit point-based formulae for the moduli (best constants) of all three Lipschitz properties in all three perturbation settings.

1. Introduction

This paper is concerned with the quantitative stability of linear inequality systems of the form

$$\{a_t' x \leq b_t,\ t \in T\}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of unknowns (understood as a column vector, with the prime denoting transposition), and $a \equiv (a_t)_{t\in T} \in (\mathbb{R}^n)^T$ and $b \equiv (b_t)_{t\in T} \in \mathbb{R}^T$ are given coefficients with T being an arbitrary (without specific topological structure) index set. The functions $t \mapsto a_t$ and $t \mapsto b_t$ are not supposed to have any particular property. When T is finite (infinite), (1) describes the feasible set of standard (semi-infinite) linear programming.

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The analysis of Lipschitz like properties (Hausdorff Lipschitz continuity, Aubin property, etc.) for finite and infinite systems (1) has a prominent and long history (e.g., [12,13,18–20,22]). In this context, particular attention has been paid to the characterization/computation of the best Lipschitz modulus, which is closely related to the so-called Hoffman constant. Very much as for simple continuity of multifunctions - which may be weakened in the two directions: upper and lower semicontinuity - the (linearly quantified) Lipschitz continuity can be split into several weakened upper and lower Lipschitz semicontinuity properties. In illustrative terms, upper semicontinuity ensures that a multifunction does not grow too fast in a neighborhood of some point of interest, whereas lower semicontinuity means that it does not suddenly collapse (in particular does not suddenly have empty values). Sometimes, some of these weakened properties (e.g. calmness and Lipschitz lower semicontinuity) are beneficially combined in order to derive important stability results under conditions that are still weaker than full Lipschitz continuity.

The topic of this work is part of variational analysis and its relation to optimization theory; the reader is addressed to the monographs [8,15,21,23]. Our quantitative stability study will be mainly focussed on the analysis of the *Lipschitz lower semicontinuity* (Lipschitz-lsc, in brief) of (1) in three perturbation settings: the context of full perturbations, i.e., simultaneous perturbations of a and b, the one of right-hand side (RHS, in brief) perturbations, where only b is perturbed, and left-hand side (LHS, in brief) perturbations, where perturbations fall only on a. The Lipschitz-lsc has been studied by many authors in different contexts (see, for example, [9,15,17,25,26]). For instance, [9, Theorem 3.8] provides some sufficient assumptions for the Lipschitz-lsc of a class of implicit multifunctions. Theorem 4.1 in [25] provides a sufficient condition for the Lipschitz-lsc of the variational system associated with a parameterized generalized equation, and [26, Remark 4] gives an upper bound of the corresponding Lipschitz-lsc modulus. Analogous results in the field of parametric constrained optimization problems are given in [26, Theorem 2]; see also [25, Proposition 6.1] and [26, Proposition 1 and Theorem 3] in relation to the argmin mapping.

As a consequence of a theorem by Klatte and Kummer (see Theorem 1 in Section 2), the Lipschitz-lsc of (1) with finite index set T is equivalent with the Aubin property (corresponding to full Lipschitz continuity) whenever both parameters a, b or just the RHS b are perturbed. The same equivalence does not hold true for only perturbations of LHS coefficients a (see Example 1 in Section 2). In order to maintain the previous equivalence under all three types of perturbation, we introduce a strengthened Lipschitz-lsc property, the so-called *Lipschitz lower semicontinuity-star* (Lipschitz-lsc*, in brief), which on the one hand comes as a natural modification of the Aubin property and which on the other hand has already been implicitly used (combined with calmness) in the past in the context of stability of parametric optimization problems [14]. Additionally, this approach has led to study, at the same time, the 'metric-regularity counterpart' of Lipschitz-lsc property (see Section 6).

Our analysis is devoted to complete the characterization of the four properties (Aubin, Lipschitz-lsc, its metric-regularity counterpart and Lipschitz-lsc*) of system (1) under all three (full, RHS, LHS) possible perturbations for a potentially infinite index set T as well as to the derivation of explicit point-based formulae for the associated moduli. The obtained results heavily rely on a previously obtained formula in [5] for the Lipschitz modulus of (1) (full perturbations). This formula has been complemented in [3] for RHS perturbations only in the context of continuous systems (where T is a compact Hausdorff space and the coefficients of the system depend continuously on the index $t \in T$). In this continuous context [7, Section 3] analyzes the lower and upper semicontinuity of the feasible set mapping under LHS perturbations (see [10, Chapter 6] for RHS and full perturbations). We also note that the moduli for the calmness of (1) (which as an upper semicontinuity property is outside the scope of the present work) have been provided in [6] in the case when T is finite under RHS and full perturbations.

2. Basic concepts, preliminary results and examples

2.1. Lipschitz properties of multifunctions

Intuitively, generalizing the well-known concept of a locally Lipschitzian function $g : Y \to X$ between metric spaces to a multifunction $\mathcal{G} : Y \rightrightarrows X$ should result in an estimate of the type

$$d\left(\mathcal{G}(y^1), \mathcal{G}(y^2)\right) \leq \kappa d\left(y^1, y^2\right), \tag{2}$$

for y^1, y^2 in a neighborhood of some fixed $\overline{y} \in Y$, where the distance between subsets of X occurring on the left-hand side has been chosen appropriately. A natural choice would be the Hausdorff distance $d := d_H$ between closed subsets of X. Writing out (2) for the Hausdorff distance between images of $\mathcal{G}$ in terms of the point-to-set distance in the given metric of X then would yield the equivalent condition

$$d\left(x, \mathcal{G}(y^2)\right) \leq \kappa d\left(y^1, y^2\right) \quad \forall x \in \mathcal{G}(y^1), \tag{3}$$

for all y^1, y^2 in a neighborhood of $\overline{y} \in Y$. This choice, however, is not very beneficial in variational analysis, where one is mostly dealing with unbounded sets (cones, systems of inequalities, etc.) which would yield the Hausdorff distance infinity and lead to an impossible estimate (2). On the other hand, one is usually interested—even for unbounded sets—just in the local behavior of multifunctions around a fixed point $(\overline{y}, \overline{x})$ of its graph, $\operatorname{gph} \mathcal{G}$, i.e., $\overline{x} \in \mathcal{G}(\overline{y})$. This suggests not to compare full images of $\mathcal{G}$ as in (2) but rather localized (around $\bar{x}$) versions thereof. Then, (3) turns into the celebrated Aubin property which is precisely defined as follows (with $\mathbb{B}_\varepsilon(y)$ referring to the closed ε-ball around some point y):

Definition 1. $\mathcal{G}$ as introduced above satisfies the Aubin property at a point $(\overline{y}, \overline{x}) \in \operatorname{gph} \mathcal{G}$ if there exist $\varepsilon, \kappa > 0$ such that

$$d\left(x, \mathcal{G}(y^2)\right) \leq \kappa d\left(y^1, y^2\right) \quad \forall x \in \mathcal{G}(y^1) \cap \mathbb{B}_\varepsilon(\overline{x}) \quad \forall y^1, y^2 \in \mathbb{B}_\varepsilon(\overline{y}). \tag{4}$$

This property has been originally introduced under the name *pseudo-Lipschitz continuity* by Aubin in [1] but later has been renamed after the author in [23]. The Aubin property plays an absolutely central role in variational analysis, be it for stability properties, for convergence of algorithms or as a constraint qualification, see, e.g., [8,15,23] and the references therein. It is closely tied with other fundamental concepts of variational analysis like *metric (sub-) regularity, (isolated, robust) calmness, error bounds, (upper, lower) Lipschitz continuity etc.* Observe that the Aubin property is both an upper and a lower Lipschitz property by allowing in (4) two parameters y^1, y^2 to vary independently around the fixed $\overline{y}$. Hence, growth or collapse of the images of $\mathcal{G}$ near $\overline{y}$ can be controlled in a linear way. If one is interested just in upper or lower Lipschitz continuity of a multifunction, then one may fix one of the parameters y^1 or y^2 as $\overline{y}$ and allow only the other parameter to vary around $\overline{y}$. For instance, fixing $y^2 := \overline{y}$ in (4), one arrives at the *calmness* property which is an upper Lipschitz property and has attracted much attention, for instance, as a constraint qualification in MPECs substantially weaker than the Aubin property, see, e.g. [11]. In this paper, we will rather focus on lower Lipschitz properties. A natural way to define one of such properties would be to fix $y^1 := \overline{y}$ and rename the variable y^2 as y in (4):

Definition 2. $\mathcal{G}$ is said to be *Lipschitz lower semicontinuous-star* (Lipschitz-lsc*, in brief) at $(\overline{y}, \overline{x}) \in \operatorname{gph} \mathcal{G}$ if there exist $\varepsilon, \kappa > 0$ such that

$$d\left(x, \mathcal{G}(y)\right) \leq \kappa d\left(y, \overline{y}\right) \quad \forall x \in \mathcal{G}(\overline{y}) \cap \mathbb{B}_\varepsilon(\overline{x}) \quad \forall y \in \mathbb{B}_\varepsilon(\overline{y}). \tag{5}$$

The asterisk on this property serves to distinguish it from the weaker concept (see below) of *Lipschitz lower semicontinuity* (Lipschitz-lsc, in brief), which has been introduced earlier (see, e.g., [15]). Indeed, the asterisk is an allusion to the *pseudo-Lipschitz** property originally introduced in [14], which combines the calmness property mentioned above and the Lipschitz-lsc* property and has been successfully applied, for instance, to the stability of probabilistic programs in [24].

Finally, we can further fix not just $y^1 := \overline{y}$ but even $x := \overline{x}$ in (4):

Definition 3. $\mathcal{G}$ is said to be *Lipschitz lower semicontinuous* at $(\overline{y}, \overline{x}) \in \operatorname{gph} \mathcal{G}$ if there exist $\varepsilon, \kappa > 0$ such that

$$d\left(\bar{x}, \mathcal{G}(y)\right) \leq \kappa d\left(y, \overline{y}\right) \quad \forall y \in \mathbb{B}_{\varepsilon}\left(\overline{y}\right). \tag{6}$$

The relevance of this last concept (also called inner calmness in [2, Definition 2.2]) is supported by the fact that it plays a crucial role in parametric optimization problems. For instance, it can be used to establish the so-called calmness from above of optimal value functions (e.g., [25, Prop. 3.2]). Moreover, it already implies the much stronger in general Aubin property for a rich class of such problems (see Theorem 1 below).

Observe that both the Lipschitz-lsc and the Lipschitz-lsc* properties—when satisfied at all $(\overline{y}, x) \in \operatorname{gph} \mathcal{G}$—imply the classical lower semicontinuity of $\mathcal{G}$ (in the sense of Berge), but differ from the latter in that they provide a linear estimate on how fast the images $\mathcal{G}(y)$ locally evade from the fixed one $\mathcal{G}(\overline{y})$.

2.2. Relations and moduli for the Aubin, Lipschitz-lsc* and Lipschitz-lsc properties

As an immediate consequence of the definitions in (4), (5) and (6), one has the implications

$$\text{Aubin property} \Longrightarrow \text{Lipschitz-lsc}^* \Longrightarrow \text{Lipschitz-lsc}, \tag{7}$$

where moreover common constants $\varepsilon, \kappa > 0$ may be used in the respective definitions. All implications are strict in general as can be seen from elementary examples. However, for certain types of multifunctions such as smooth, fully parameterized inequality systems, all three concepts may coincide:

Theorem 1 *([16, Lemma 1]). For $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^p)$, let $\mathcal{G} : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ be defined as $\mathcal{G}(y) := \{x \in \mathbb{R}^n | g(x) \leq y\}$. Moreover, for some $\overline{x} \in \mathbb{R}^n$ assume that $g(\bar{x}) \leq 0$. Then, $\mathcal{G}$ has the Aubin property at $(0, \overline{x})$ if and only if $\mathcal{G}$ is Lipschitz lsc at $(0, \overline{x})$ (hence, if and only if $\mathcal{G}$ is Lipschitz lsc* at $(0, \overline{x})$).*

This result hinges essentially on the fact that the inequality system is fully perturbed (includes RHS perturbations). This can be seen from the following simple example of a single non-fully perturbed linear inequality:

Example 1. Let $\mathcal{G}(y) := \{x \in \mathbb{R} \mid y \cdot x \leq 0\}$ for $y \in \mathbb{R}$. Then,

$$\mathcal{G}(y) = \begin{cases} \mathbb{R}_+ & y < 0, \\ \mathbb{R} & y = 0, \\ \mathbb{R}_- & y > 0, \end{cases}$$

is Lipschitz-lsc at $(0, 0)$ but it is not Lipschitz-lsc* there. Due to (7), it also fails to have the Aubin property at $(0, 0)$.

Motivated by this example, part of our work will be devoted to the relation of the three properties in the case of just LHS perturbations of linear inequality systems (coefficients). As a consequence, we will derive that in the finite case the Aubin property is still equivalent with the Lipschitz-lsc*.

From the qualitative comparison of the three properties, one may pass to a stronger quantitative comparison of their associated moduli which are defined as the smallest $\kappa > 0$ such that for some $\varepsilon > 0$ the respective inequality holds true:

$$\operatorname{lip} \mathcal{G}(\overline{y}, \overline{x}) \,/\, \operatorname{liplsc}^* \mathcal{G}(\overline{y}, \overline{x}) \,/\, \operatorname{liplsc} \mathcal{G}(\overline{y}, \overline{x}) := \inf \{\kappa > 0 | \exists \varepsilon > 0 : (4)/(5)/(6)\} \tag{8}$$

By definition, the Aubin property (Lipschitz-lsc*, Lipschitz-lsc property) holds true if and only if

$$\operatorname{lip} \mathcal{G}(\overline{y}, \overline{x}) < \infty \quad (\operatorname{liplsc}^* \mathcal{G}(\overline{y}, \overline{x}) < \infty, \operatorname{liplsc} \mathcal{G}(\overline{y}, \overline{x}) < \infty)\,. \tag{9}$$

Consequently, calculating the moduli—as we shall do in the following—will immediately imply results on satisfaction/non-satisfaction of the corresponding properties. Moreover, the implications (7) holding true with common constants $\varepsilon, \kappa > 0$ in the respective definitions yield the inequalities

$$\operatorname{lip} \mathcal{G}(\overline{y}, \overline{x}) \geq \operatorname{liplsc}^* \mathcal{G}(\overline{y}, \overline{x}) \geq \operatorname{liplsc} \mathcal{G}(\overline{y}, \overline{x}). \tag{10}$$

However, for a class of multifunctions where all three properties are equivalent—as in Theorem 1—it is not automatically clear that inequalities (10) can be reverted, thus establishing equality of all moduli. For instance, if $\varepsilon, \kappa > 0$ are such that (6) is satisfied, then it is not guaranteed that the same $\kappa > 0$ would work in the estimate (4) (possibly with some $\tilde{\varepsilon}$ different from ε) even if one knew that (4) holds true for some $\tilde{\varepsilon}, \tilde{\kappa} > 0$. Therefore, the equality of moduli—which we will basically establish for the class of linear inequality systems (1)—is a much stronger result than the equivalence of the properties themselves.

It is easy to see that the moduli defined in (8) can be equivalently transformed into the more handy expressions:

$$\operatorname{lip} \mathcal{G}(\overline{y}, \overline{x}) = \limsup_{\substack{(y^1, y^2, x) \to (\overline{y}, \overline{y}, \overline{x}) \\ y^1 \neq y^2,\, x \in \mathcal{G}(y^1)}} \frac{d\left(x, \mathcal{G}(y^2)\right)}{d\left(y^1, y^2\right)}, \tag{11}$$

$$\operatorname{liplsc}^* \mathcal{G}(\overline{y}, \overline{x}) = \limsup_{\substack{(y, x) \to (\overline{y}, \overline{x}) \\ y \neq \overline{y},\, x \in \mathcal{G}(\overline{y})}} \frac{d\left(x, \mathcal{G}(y)\right)}{d\left(\overline{y}, y\right)}, \tag{12}$$

$$\operatorname{liplsc} \mathcal{G}(\overline{y}, \overline{x}) = \limsup_{y \to \overline{y},\, y \neq \overline{y}} \frac{d\left(\bar{x}, \mathcal{G}(y)\right)}{d\left(\overline{y}, y\right)}. \tag{13}$$

The following formula for the Lipschitz modulus, sometimes easier to handle, is well-known and relies on the fact that the Aubin property of a mapping is equivalent with the metric regularity of its inverse (see, e.g. [8,15]):

$$\operatorname{lip} \mathcal{G}(\overline{y}, \overline{x}) = \limsup_{(y,x) \to (\overline{y}, \overline{x}), y \notin \mathcal{G}^{-1}(x)} \frac{d\left(x, \mathcal{G}(y)\right)}{d\left(y, \mathcal{G}^{-1}(x)\right)}, \tag{14}$$

Observe, that if $(\overline{y}, \overline{x})$ is an interior point of $\operatorname{gph} \mathcal{G}$, then $\operatorname{lip}\mathcal{G}(\overline{y}, \overline{x}) = 0$ by (11). On the other hand, if the limsup in (14) is taken over the empty set, hence formally yields the value zero too. The following example shows that the inequalities in (10) may be strict, even if the moduli are finite (i.e., if all three properties hold true):

Example 2. Let $\mathcal{G} : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $\mathcal{G}(y) := \{0\} \cup [|y|, |y| + 1]$. We want to calculate the moduli (8) at the point $(\bar{y}, \bar{x}) := (0, 0) \in \operatorname{gph} \mathcal{G}$. Observe first that $0 \in \mathcal{G}(y)$ for all y, whence trivially $\operatorname{liplsc} \mathcal{G}(0, 0) = 0$ by (13). Next, in order to calculate $\operatorname{liplsc}^* \mathcal{G}(0, 0)$ via (12), consider sequences $(y_k, x_k) \to (0, 0)$ such that $y_k \neq 0$ and $x_k \in \mathcal{G}(0) = [0, 1]$. It is easily verified that

$$\frac{d(x_k, \mathcal{G}(y_k))}{d(y_k, 0)} = \frac{\min\{x_k, \max\{|y_k| - x_k, 0\}\}}{|y_k|} \leq 1/2,$$

where equality is realized for the concrete sequence $y_k := k^{-1}, x_k := k^{-1}/2$. Hence, the lim sup in (12) equals 1/2, showing that $\operatorname{liplsc}^* \mathcal{G}(0,0) = 1/2$. Finally, for the computation of $\operatorname{lip} \mathcal{G}(0,0)$, observe that for any $y^1, y^2 \in \mathbb{R}$, and any $x \in \mathcal{G}\left(y^1\right)$ we have that

$$d\left(x, \mathcal{G}\left(y^2\right)\right) = \begin{cases} \min\{x, \left|y^2\right| - x\} & \text{if } x \leq \left|y^2\right| \\ 0 & \text{if } \left|y^2\right| \leq x \leq \left|y^2\right| + 1 \\ x - \left(\left|y^2\right| + 1\right) & \text{if } x > \left|y^2\right| + 1 \end{cases}.$$

In any case, taking into account that either $x = 0$ or $\left|y^1\right| \leq x \leq \left|y^1\right| + 1$, it follows that

$$d\left(x, \mathcal{G}\left(y^2\right)\right) \leq \left|\left|y^1\right| - \left|y^2\right|\right| \leq \left|y^1 - y^2\right|,$$

whence $\operatorname{lip} \mathcal{G}(\overline{y}, \overline{x}) \leq 1$ by (11). On the other hand, for the particular sequence

$$\left(y_k^1, y_k^2, x_k\right) := \left(k^{-1}, 2k^{-1}, k^{-1}\right) \to (0,0,0),$$

one has that $y_k^1 \neq y_k^2$ and

$$d\left(x_k, \mathcal{G}\left(y_k^2\right)\right) = k^{-1} = \left|y_k^1 - y_k^2\right|,$$

whence $\operatorname{lip} \mathcal{G}(\overline{y}, \overline{x}) = 1$.

2.3. Perturbation settings and preparatory results

To start with, we introduce some notation. Given $X \subset \mathbb{R}^p$, we denote by $\operatorname{conv} X$ and $\operatorname{cone} X$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that $\operatorname{cone} X$ always contains the zero vector 0_p and, hence, $\operatorname{cone} \emptyset = \{0_p\}$. Moreover, throughout this paper we use the conventions

$$\frac{0}{0} := 0, \ \frac{1}{0} := \infty, \ \text{and} \ \frac{1}{\infty} := 0. \tag{15}$$

The interior, closure and boundary of X are denoted by $\operatorname{int} X$, $\operatorname{cl} X$ and $\operatorname{bd} X$, respectively.

We define our parameter space as $\Theta := \left(\mathbb{R}^n \times \mathbb{R}\right)^T$. Associated with systems of the form (1) in the setting of full perturbations, we consider the feasible set mapping $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{F}(\sigma) := \left\{x \in \mathbb{R}^n \mid a_t' x \leq b_t \quad \forall t \in T\right\}, \ \sigma \equiv (a, b) \in \Theta. \tag{16}$$

When we fix $a = \overline{a}$ and $b = \overline{b}$ separately, we deal with the contexts of RHS and LHS perturbations, respectively, and the corresponding partial feasible set mappings,

$$\mathcal{F}_{\overline{a}} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n \text{ and } \mathcal{F}_{\overline{b}} : \left(\mathbb{R}^n\right)^T \rightrightarrows \mathbb{R}^n,$$

are defined by

$$\mathcal{F}_{\overline{a}}(b) := \mathcal{F}(\overline{a}, b) \text{ and } \mathcal{F}_{\overline{b}}(a) := \mathcal{F}(a, \overline{b}).$$

Along this work, the space of variables $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, whose *dual norm* is, as usual, denoted by $\|\cdot\|_*$ and defined as

$$\|u\|_* = \max_{\|x\| \leq 1} |u'x| .$$

By d_* we represent the distance in $\mathbb{R}^n$ associated with $\|\cdot\|_*$. The parameter space Θ is endowed with the *extended distance* $d : \Theta \times \Theta \to [0, \infty]$ given by

$$d(\sigma^1, \sigma^2) := \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t^2 \\ b_t^2 \end{pmatrix} \right\| ,$$

where $\|\cdot\|$ is the norm in $\mathbb{R}^{n+1}$ defined as

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| := \max \{\|u\|_*, |v|\} \text{ for all } \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{n+1}, \tag{17}$$

whose dual norm is given by

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_* = \|u\| + |v|, \text{ whenever } \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{n+1}.$$

In the parameter spaces $\mathbb{R}^T$ and $(\mathbb{R}^n)^T$, associated with partial perturbations, we use the natural extended distances (all denoted by d and distinguished by the arguments):

$$d(b^1, b^2) := \sup_{t \in T} \left| b_t^1 - b_t^2 \right| , \; b^1, b^2 \in \mathbb{R}^T,$$
$$d(a^1, a^2) := \sup_{t \in T} \left\| a_t^1 - a_t^2 \right\|_* , \; a^1, a^2 \in (\mathbb{R}^n)^T .$$

We recall the well-known Farkas Lemma which can be traced out from [10, Corollary 3.1.2] and constitutes a key tool in the paper.

Lemma 1 *(Extended Farkas Lemma). The inequality $a'x \leq b$ is a consequence of a consistent system $a_t'x \leq b_t \quad (t \in T)$ if and only if*

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \text{cl cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T ; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}.$$

In the following theorem we gather well-known results characterizing the Aubin property for $\mathcal{F}$ at $(\overline{\sigma}, \overline{x}) \in \text{gph}\mathcal{F}$. We appeal to the well-known notion of a *strong Slater constraint qualification* (SSCQ, in brief) which is satisfied at σ if there exists $\widehat{x} \in \mathbb{R}^n$ (called a *strong Slater element*, SS element in brief) and a positive scalar ε such that $a_t'\widehat{x} \leq b_t - \varepsilon$ for all $t \in T$. The equivalences '$(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$' can be found in [10, Theorem 6.1]. Obviously, '$(i) \Rightarrow (ii)$' and, from [4, Corollary 5], we obtain the converse implication. For convenience, we have also added the trivial equivalence '$(iv) \Leftrightarrow (v)$', where we appeal to the closed and convex set

$$C_{\overline{x}} := \left\{ u \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} u \\ u'\overline{x} \end{pmatrix} \in \text{cl conv}(\overline{\sigma}) \right\}, \tag{18}$$

where $\text{conv}(\sigma)$ stands for $\text{conv}\left\{ \binom{a_t}{b_t}, \; t \in T \right\}$ whenever $\sigma \equiv (a, b)$.

Theorem 2. *Let $(\overline{\sigma}, \overline{x}) \in \text{gph}\mathcal{F}$. The following statements are equivalent:*

(i) $\mathcal{F}$ satisfies the Aubin property at $(\overline{\sigma}, \overline{x})$;

(ii) $\mathcal{F}(\sigma) \neq \emptyset$ for all σ in some neighborhood of $\overline{\sigma}$;

(iii) SSCQ is satisfied at $\overline{\sigma}$*;*
(iv) $0_{n+1} \notin \operatorname{cl}\operatorname{conv}(\overline{\sigma})$*;*
(v) $0_n \notin C_{\overline{x}}$*.*

The following theorem constitutes a key starting point in the remaining sections. Observe that, in the particular case when T is finite, a standard argument yields that $C_{\overline{x}} = \operatorname{conv}\{\overline{a}_t, t \in T_{\overline{\sigma}}(\overline{x})\}$, where $T_{\overline{\sigma}}(\overline{x})$ is the set of active indices of $\overline{\sigma}$ at $\overline{x} \in \mathcal{F}(\overline{\sigma})$, i.e.,

$$T_{\overline{\sigma}}(\overline{x}) := \{t \in T \mid \overline{a}_t'\overline{x} = \overline{b}_t\}.$$

Theorem 3 *([5, Theorem 1]). Assume that* $\{\overline{a}_t, t \in T\}$ *is bounded and let* $(\overline{\sigma}, \overline{x}) \in \operatorname{gph}\mathcal{F}$*. One has that*

$$\operatorname{lip}\mathcal{F}(\overline{\sigma}, \overline{x}) = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, C_{\overline{x}}\right)}.$$

Consequently, when T *is finite, this reduces to*

$$\frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{conv}\{\overline{a}_t, t \in T_{\overline{\sigma}}(\overline{x})\}\right)}.$$

To finish this subsection, we include two lemmas which are technical results needed later on in our derivation, where $[\alpha]_+ := \max\{0, \alpha\}$:

Lemma 2 *([5, Lemma 1]). Let* $\sigma \in \operatorname{dom}\mathcal{F}$ *and* $z \in \mathbb{R}^n$*. Then, we have:*

$$d\left(z, \mathcal{F}(\sigma)\right) = \sup_{\binom{u}{v} \in \operatorname{conv}(\sigma)} \frac{[u'z - v]_+}{\|u\|_*}\ .$$

Lemma 3. *Let* $\sigma \equiv (a, b) \in (\mathbb{R}^n \times \mathbb{R})^T$ *and* $z \in \mathbb{R}^n$*. Then, we have that:*
(i) $d\left(\sigma, \mathcal{F}^{-1}(z)\right) = \dfrac{\sup_{t\in T}[a_t'z - b_t]_+}{\|z\| + 1}$*;*
(ii) $d\left(b, \mathcal{F}_{\overline{a}}^{-1}(z)\right) = \sup_{t\in T}[\overline{a}_t'z - b_t]_+$*;*
(iii) $d\left(a, \mathcal{F}_{\overline{b}}^{-1}(z)\right) = \dfrac{\sup_{t\in T}[a_t'z - \overline{b}_t]_+}{\|z\|}$*.*

Proof. (i) has been established in [4, Lemma 10].

(ii) follows a standard argument, which we summarize here for completeness: given any $b \in \mathbb{R}^T$ and $z \in \mathbb{R}^n$, define $\widetilde{b} \in \mathbb{R}^T$ by

$$\widetilde{b}_t := b_t + [\overline{a}_t'z - b_t]_+,\ t \in T;$$

obviously, $d\left(b, \widetilde{b}\right) = \sup_{t\in T}[\overline{a}_t'z - b_t]_+$ and $\widetilde{b} \in \mathcal{F}_{\overline{a}}^{-1}(z)$. Moreover, one easily sees that there is no other $\widehat{b} \in \mathcal{F}_{\overline{a}}^{-1}(z)$ such that $d\left(b, \widehat{b}\right) < d\left(b, \widetilde{b}\right)$.

To verify (iii), first, in the particular case $z = 0_n$, we have

$$\mathcal{F}_{\overline{b}}^{-1}(0_n) = \begin{cases} (\mathbb{R}^n)^T, & \text{if } \overline{b}_t \geq 0,\ \forall t \in T \\ \emptyset, & \text{otherwise.} \end{cases}$$

So, $d\left(a, \mathcal{F}_{\overline{b}}^{-1}(0_n)\right) = 0$, if $\overline{b}_t \geq 0$, $t \in T$, and $d\left(a, \mathcal{F}_{\overline{b}}^{-1}(0_n)\right) = \infty$ otherwise. In this way, applying our conventions (15), (iii) trivially holds true. Assume now that $z \neq 0_n$. Take any $a \in (\mathbb{R}^n)^T$ and $z \in \mathbb{R}^n$. Let $u \in \mathbb{R}^n$ such that $\|u\|_* = 1$ and $u'z = \|z\|$. Define $\widetilde{a} \in (\mathbb{R}^n)^T$ by

$$\widetilde{a}_t = a_t - \frac{[a_t'z - \overline{b}_t]_+}{\|z\|} u, \ t \in T.$$

Obviously, $\widetilde{a} \in \mathcal{F}_{\overline{b}}^{-1}(z)$ since

$$\widetilde{a}_t'z - \overline{b}_t = a_t'z - [a_t'z - \overline{b}_t]_+ - \overline{b}_t \leq 0, \ t \in T.$$

Hence,

$$d\left(a, \mathcal{F}_{\overline{b}}^{-1}(z)\right) \leq d(a, \widetilde{a}) = \frac{\sup_{t \in T} [a_t'z - \overline{b}_t]_+}{\|z\|}.$$

Now, arguing by contradiction, assume the existence of $\widehat{a} \in \mathcal{F}_{\overline{b}}^{-1}(z)$ such $d(a, \widehat{a}) < d(a, \widetilde{a})$. Then, there exists $t_0 \in T$ such that

$$\|a_{t_0} - \widehat{a}_{t_0}\|_* < \|a_{t_0} - \widetilde{a}_{t_0}\|_* = \frac{[a_{t_0}'z - \overline{b}_{t_0}]_+}{\|z\|}. \tag{19}$$

From (19), we deduce that $[a_{t_0}'z - \overline{b}_{t_0}]_+ > 0$, and so a_{t_0} does not belong to the half space $H := \{u \in \mathbb{R}^n \mid z'u \leq \overline{b}_{t_0}\}$. Now, the well-known Ascoli formula for the distance from a point to a hyperplane, yields the following contradiction with (19):

$$\frac{[a_{t_0}'z - \overline{b}_{t_0}]_+}{\|z\|} = d_*(a_{t_0}, H) \leq \|a_{t_0} - \widehat{a}_{t_0}\|_*. \quad \square$$

3. Lower Lipschitz moduli under full perturbations

This section is concerned with the Lipschitz-lsc and Lipschitz-lsc* moduli of the feasible set mapping $\mathcal{F}$ in the framework of perturbations of all coefficients. To compute the desired moduli we refer in a first step to the feasible set mapping associated with just a single inequality, $\mathcal{L} : \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$, defined as

$$\mathcal{L}\begin{pmatrix} a \\ b \end{pmatrix} := \{x \in \mathbb{R}^n \mid a'x \leq b\}\ .$$

The following lemma provides the Lipschitz-lsc modulus of $\mathcal{L}$, which indeed coincides with the Lipschitz modulus.

Lemma 4. *Let $\overline{u} \neq 0_n$. Then, for any $\overline{x}$ one has that*

$$\operatorname{liplsc} \mathcal{L}\left(\begin{pmatrix} \overline{u} \\ \overline{u}'\overline{x} \end{pmatrix}, \overline{x}\right) = \frac{\|\overline{x}\| + 1}{\|\overline{u}\|_*} = \operatorname{lip} \mathcal{L}\left(\begin{pmatrix} \overline{u} \\ \overline{u}'\overline{x} \end{pmatrix}, \overline{x}\right).$$

Proof. The second equality in the statement above follows from Theorem 3. So, it remains to prove the first one.

Appealing to (13), and applying the Ascoli formula, we have that

$$\begin{aligned}\operatorname{liplsc}\mathcal{L}\left(\begin{pmatrix}\overline{u}\\ \overline{u}'\overline{x}\end{pmatrix},\overline{x}\right) &= \limsup_{\binom{a}{b}\to\binom{\overline{u}}{\overline{u}'\overline{x}},\binom{a}{b}\neq\binom{\overline{u}}{\overline{u}'\overline{x}}} \frac{d\left(\overline{x},\mathcal{L}\binom{a}{b}\right)}{\left\|\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right\|} \\ &= \limsup_{\binom{a}{b}\to\binom{\overline{u}}{\overline{u}'\overline{x}},\binom{a}{b}\neq\binom{\overline{u}}{\overline{u}'\overline{x}}} \frac{1}{\|a\|_*}\frac{[a'\overline{x}-b]_+}{\left\|\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right\|} \\ &= \max\left\{0, \limsup_{\binom{a}{b}\to\binom{\overline{u}}{\overline{u}'\overline{x}},\binom{a}{b}\neq\binom{\overline{u}}{\overline{u}'\overline{x}},a'\overline{x}\geq b} \frac{1}{\|a\|_*}\frac{a'\overline{x}-b}{\left\|\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right\|}\right\}.\end{aligned}$$

In the third equality, we have split the limsup according to the two cases $a'\overline{x} < b$ and $a'\overline{x} \geq b$. On the other hand, we have that

$$\limsup_{\binom{a}{b}\to\binom{\overline{u}}{\overline{u}'\overline{x}},\binom{a}{b}\neq\binom{\overline{u}}{\overline{u}'\overline{x}}} \frac{1}{\|a\|_*}\frac{a'\overline{x}-b}{\left\|\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right\|} = \limsup_{\binom{a}{b}\to\binom{\overline{u}}{\overline{u}'\overline{x}},\binom{a}{b}\neq\binom{\overline{u}}{\overline{u}'\overline{x}},a'\overline{x}\geq b} \frac{1}{\|a\|_*}\frac{a'\overline{x}-b}{\left\|\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right\|} \geq 0.$$

Here, the $\geq$ part of the equality is evident by omitting the constraint $a'\overline{x} \geq b$ in the limsup and the $\leq$ part follows from the fact that the corresponding limsup subject to the opposite constraint $a'\overline{x} < b$ is always nonpositive. Hence, upon subtracting an artificial zero, we may proceed as

$$\operatorname{liplsc}\mathcal{L}\left(\begin{pmatrix}\overline{u}\\ \overline{u}'\overline{x}\end{pmatrix},\overline{x}\right) = \limsup_{\binom{a}{b}\to\binom{\overline{u}}{\overline{u}'\overline{x}},\binom{a}{b}\neq\binom{\overline{u}}{\overline{u}'\overline{x}}} \frac{1}{\|a\|_*}\frac{\left(\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right)'\binom{\overline{x}}{-1}}{\left\|\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right\|} = \frac{1}{\|\overline{u}\|_*}\left\|\begin{pmatrix}\overline{x}\\ -1\end{pmatrix}\right\|_*,$$

where the last equality comes from the fact that the expression

$$\frac{\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}}{\left\|\binom{a}{b}-\binom{\overline{u}}{\overline{u}'\overline{x}}\right\|},$$

may be any vector of the unit sphere of $\mathbb{R}^{n+1}$. Finally, recall that for our choice of norms $\left\|\binom{\overline{x}}{-1}\right\|_* = \|\overline{x}\| + 1$. □

We are now in a position to formulate our first result on the equality of all three considered Lipschitz moduli (11, 12, 13) of the feasible set mapping (16) in case that the coefficients $\{\overline{a}_t, t \in T\}$ are bounded (and in particular if T is finite).

Theorem 4. *Assume that $\{\overline{a}_t, t \in T\}$ is bounded and let $(\overline{\sigma}, \overline{x}) \in \operatorname{gph}\mathcal{F}$. Then,*

$$\operatorname{liplsc}\mathcal{F}(\overline{\sigma},\overline{x}) = \operatorname{liplsc}^*\mathcal{F}(\overline{\sigma},\overline{x}) = \operatorname{lip}\mathcal{F}(\overline{\sigma},\overline{x}) = \frac{\|\overline{x}\|+1}{d_*\left(0_n, C_{\overline{x}}\right)}, \tag{20}$$

where all quantities reduce to

$$\frac{\|\overline{x}\|+1}{d_*\left(0_n, \operatorname{conv}\{\overline{a}_t, t\in T_{\overline{\sigma}}(\overline{x})\}\right)},$$

if T even happens to be a finite set.

Proof. By virtue of Theorem 3 and (10) it will be sufficient to lead the following relation to a contradiction:

$$\operatorname{liplsc} \mathcal{F}(\overline{\sigma}, \overline{x}) < \frac{\|\overline{x}\| + 1}{d_* \left(0_n, C_{\overline{x}}\right)}. \tag{21}$$

First, observe that if $0_n \in C_{\overline{x}}$, then $\mathcal{F}$ fails to be Lipschitz-lsc at $(\overline{\sigma}, \overline{x})$ by Theorem 2. Consequently, $\operatorname{liplsc} \mathcal{F}(\overline{\sigma}, \overline{x}) = \infty$ yielding a contradiction with (21). So, from now on, we assume $0_n \notin C_{\overline{x}}$, and then $d_* \left(0_n, C_{\overline{x}}\right) = \|\overline{u}\|_*$ for some $0_n \neq \overline{u} \in C_{\overline{x}}$. If (21) were true, then there existed some $\alpha > 0$ such that

$$\operatorname{liplsc} \mathcal{F}(\overline{\sigma}, \overline{x}) < \alpha < \frac{\|\overline{x}\| + 1}{d_* \left(0_n, C_{\overline{x}}\right)} = \frac{\|\overline{x}\| + 1}{\|\overline{u}\|_*} = \operatorname{liplsc} \mathcal{L}\left(\begin{pmatrix} \overline{u} \\ \overline{u}'\overline{x} \end{pmatrix}, \overline{x}\right), \tag{22}$$

where the last equality follows from Lemma 4. Hence, there is a sequence $(u^r, v^r) \to_r (\overline{u}, \overline{u}'\overline{x})$ such that for all r one has that $(u^r, v^r) \neq (\overline{u}, \overline{u}'\overline{x})$ and

$$d\left(\overline{x}, \mathcal{L}\begin{pmatrix} u^r \\ v^r \end{pmatrix}\right) > \alpha d\left(\begin{pmatrix} u^r \\ v^r \end{pmatrix}, \begin{pmatrix} \overline{u} \\ \overline{u}'\overline{x} \end{pmatrix}\right) .$$

Define a sequence $\sigma^r \equiv (a^r, b^r)$ componentwise by

$$a_t^r := \overline{a}_t + u^r - \overline{u}; \quad b_t^r := \overline{b}_t + v^r - \overline{u}'\overline{x} \ \forall t \in T.$$

Clearly $\sigma^r \to_r \bar{\sigma} \equiv \left(\bar{a}, \bar{b}\right)$. Since $\overline{u} \in C_{\overline{x}}$, we can write

$$\begin{pmatrix} \overline{u} \\ \overline{u}'\overline{x} \end{pmatrix} = \lim_k \sum_{t \in T} \lambda_t^k \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix},$$

for certain $\lambda^k \equiv \left(\lambda_t^k\right)_{t \in T} \in \mathbb{R}_+^{(T)}$ (i.e., for each $k \in \mathbb{N}$, $\lambda_t^k = 0$ except for finitely many $t \in T$) such that $\sum_{t \in T} \lambda_t^k = 1$ for each k. Then,

$$\lim_k \sum_{t \in T} \lambda_t^k \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} = \begin{pmatrix} u^r \\ v^r \end{pmatrix} \quad \forall r \in \mathbb{N}.$$

This entails that

$$\begin{pmatrix} u^r \\ v^r \end{pmatrix} \in \operatorname{cl} \operatorname{cone} \left\{ \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}.$$

Moreover, $0_n \notin C_{\overline{x}}$ implies via Theorem 2 that $\mathcal{F}(\sigma) \neq \emptyset$ for σ in some neighborhood of $\bar{\sigma}$. Hence, the system $\left\{x \in \mathbb{R}^n \mid (a_t^r)' x \leq b_t^r, \ t \in T\right\}$ is consistent for all r large enough. Now, Lemma 1 provides that $\mathcal{F}(\sigma^r) \subseteq \mathcal{L}\binom{u^r}{v^r}$, whence

$$d\left(\overline{x}, \mathcal{F}(\sigma^r)\right) \geq d\left(\overline{x}, \mathcal{L}\begin{pmatrix} u^r \\ v^r \end{pmatrix}\right) > \alpha d\left(\begin{pmatrix} u^r \\ v^r \end{pmatrix}, \begin{pmatrix} \overline{u} \\ \overline{u}'\overline{x} \end{pmatrix}\right) = \alpha d\left(\sigma^r, \bar{\sigma}\right),$$

for r large enough. We arrive at the contradiction $\operatorname{liplsc} \mathcal{F}(\overline{\sigma}, \overline{x}) \geq \alpha$ with (22). This finally disproves (21). □

4. Lower Lipschitz moduli under RHS perturbations

In this section, we redo the previous analysis but for the partial mapping $\mathcal{F}_{\overline{a}}$ where just the RHS b of system (1) is varied. Again, the first step of this section consists in the computation of the Lipschitz-lsc moduli for a (partially) perturbed single inequality, namely the mapping $\mathcal{L}_{\overline{a}} : \mathbb{R} \rightrightarrows \mathbb{R}^n$ defined by

$$\mathcal{L}_{\overline{a}}(b) := \{ x \in \mathbb{R}^n \,|\, \overline{a}' x \leq b \} .$$

One immediately derives the following relation for any $\overline{u} \neq 0_n$ and $\overline{x}$, one has that

$$\operatorname{liplsc} \mathcal{L}_{\overline{u}} \left(\overline{u}' \overline{x}, \overline{x} \right) = \limsup_{b \to \overline{u}' \overline{x}, b \neq \overline{u}' \overline{x}} \frac{d \left(\overline{x}, \mathcal{L}_{\overline{u}}(b) \right)}{|\overline{u}' \overline{x} - b|} = \frac{1}{\|\overline{u}\|_*} \limsup_{b \to \overline{u}' \overline{x}, b \neq \overline{u}' \overline{x}} \frac{[\overline{u}' \overline{x} - b]_+}{|\overline{u}' \overline{x} - b|} = \frac{1}{\|\overline{u}\|_*}. \tag{23}$$

This allows us to obtain the analogous result of Theorem 4, this time for the partial mapping $\mathcal{F}_{\overline{a}}$ and with a modified formula for the moduli:

Theorem 5. *Assume that* $\{\overline{a}_t, t \in T\}$ *is bounded and let* $\overline{x} \in \mathcal{F}(\overline{\sigma})$. *Then,*

$$\operatorname{liplsc} \mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \operatorname{liplsc}^* \mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \operatorname{lip} \mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \frac{1}{d_* \left(0_n, C_{\overline{x}} \right)},$$

where all quantities reduce to

$$\frac{1}{d_* \left(0_n, \operatorname{conv} \{ \overline{a}_t, t \in T_{\overline{\sigma}}(\overline{x}) \} \right)},$$

if T even happens to be a finite set.

Proof. First observe that Lemma 3 (i) (applied to the special element $\sigma := (\overline{a}, b)$) and (ii) yield the following relation for all x and b:

$$d \left(b, \mathcal{F}_{\overline{a}}^{-1}(x) \right) = (\|x\| + 1) \, d \left((\overline{a}, b), \mathcal{F}^{-1}(x) \right) .$$

Now, by virtue of (14),

$$\begin{aligned}
\operatorname{lip} \mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) &= \limsup_{(x,b) \to (\overline{x}, \overline{b}), b \notin \mathcal{F}_{\overline{a}}^{-1}(x)} \frac{d \left(x, \mathcal{F}_{\overline{a}}(b) \right)}{d \left(b, \mathcal{F}_{\overline{a}}^{-1}(x) \right)} \\
&= \limsup_{(x,b) \to (\overline{x}, \overline{b}), b \notin \mathcal{F}_{\overline{a}}^{-1}(x)} \frac{d \left(x, \mathcal{F}(\overline{a}, b) \right)}{(\|x\| + 1) \, d \left((\overline{a}, b), \mathcal{F}^{-1}(x) \right)} \\
&\leq \frac{1}{\|\overline{x}\| + 1} \limsup_{(x,\sigma) \to (\overline{x}, \overline{\sigma}), \sigma \notin \mathcal{F}^{-1}(x)} \frac{d \left(x, \mathcal{F}(\sigma) \right)}{d \left(\sigma, \mathcal{F}^{-1}(x) \right)} \\
&= \frac{1}{\|\overline{x}\| + 1} \operatorname{lip} \mathcal{F}(\overline{\sigma}, \overline{x}) = \frac{1}{d_* \left(0_n, C_{\overline{x}} \right)},
\end{aligned}$$

where the last equality follows from Theorem 4. With the same argument as in the proof of Theorem 4, it will be sufficient to lead the relation

$$\operatorname{liplsc} \mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) < \frac{1}{d_* \left(0_n, C_{\overline{x}} \right)}, \tag{24}$$

to a contradiction. If $0_n \in C_{\overline{x}}$, we have $\overline{b} \notin \operatorname{intdom} \mathcal{F}_{\overline{a}}$ (see Theorem 2), which implies $\operatorname{liplsc} \mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \infty$, which contradicts (24). So, from now on, we assume $0_n \notin C_{\overline{x}}$, and again let $0_n \neq \overline{u} \in C_{\overline{x}}$ such that $d_* (0_n, C_{\overline{x}}) = \|\overline{u}\|_*$. By (24), there exists $\alpha > 0$ such that (see (23))

$$\operatorname{liplsc} \mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) < \alpha < \frac{1}{d_* (0_n, C_{\overline{x}})} = \frac{1}{\|\overline{u}\|_*} = \operatorname{liplsc} \mathcal{L}_{\overline{u}}(\overline{u}'\overline{x}, \overline{x}). \tag{25}$$

Hence, there exists a sequence $\{v^r\}$ converging to $\overline{u}'\overline{x}$ such that

$$d(\overline{x}, \mathcal{L}_{\overline{u}}(v^r)) > \alpha |v^r - \overline{u}'\overline{x}|, \; r = 1, 2, ... \tag{26}$$

The rest of the proof follows a similar argument to the one of Theorem 4, by considering the restricted sequence $\{(\overline{a}, b^r)\}_{r\in\mathbb{N}}$ with the same definition $b_t^r := \overline{b}_t + v^r - \overline{u}'\overline{x}$. As in the proof of Theorem 4, $0_n \notin C_{\overline{x}}$ implies that $\mathcal{F}(\sigma) \neq \emptyset$ for σ locally around $\bar{\sigma}$. Hence, the (reduced) system $\{x \in \mathbb{R}^n \mid \overline{a}_t' x \leq b_t^r, \; t \in T\}$ is consistent for all r large enough and Lemma 1 provides that $\mathcal{F}(\overline{a}, b^r) \subseteq \mathcal{L}_{\overline{u}}(v^r)$. Consequently,

$$d(\overline{x}, \mathcal{F}(\overline{a}, b^r)) \geq d(\overline{x}, \mathcal{L}_{\overline{u}}(v^r)) \text{ and } d\left(b^r, \overline{b}\right) = |v^r - \overline{u}'\overline{x}|,$$

for r sufficiently large. So, appealing to (26) we derive

$$d(\overline{x}, \mathcal{F}((\overline{a}, b^r))) \geq \alpha |v^r - \overline{u}'\overline{x}|,$$

for r large enough, which represents a contradiction with the first inequality of (25). □

5. Lower Lipschitz moduli under LHS perturbations

The main objective of this section is to compute the Lipschitz, Lipschitz-lsc, and Lipschitz-lsc* moduli of $\mathcal{F}_{\overline{b}}$ at $(\overline{a}, \overline{x}) \in \operatorname{gph}\mathcal{F}_{\overline{b}}$. To start with, we emphasize that the fact of considering only LHS perturbations entails notable differences with respect to the previous frameworks, where the three moduli coincide. We recall Example 1, representing a single linear inequality with LHS perturbations only. As we have seen before, the associated feasible set mapping is Lipschitz-lsc (actually with modulus zero), while it fails to be Lipschitz-lsc* and much less to have the Aubin property (hence, both moduli equal infinity). The following example shows that the three moduli can coincide and be finite in spite of the failure of SSCQ which is in clear contrast with the corresponding results for full or RHS perturbations (see previous results and Theorem 2).

Example 3. Let $\overline{b} := 0_3$ and consider the mapping $\mathcal{F}_{\overline{b}}(a)$ for a close to $\bar{a}$ with

$$\bar{a}_1 := (-1, 1)', \quad \bar{a}_2 := (1, 1)', \quad \bar{a}_3 := (0, -1)',$$

and $\bar{x} := 0_2 \in \mathcal{F}_{\overline{b}}(\bar{a})$. One easily checks that SSCQ fails at $\bar{\sigma} := (\bar{a}, \overline{b})$. On the other hand, $\mathcal{F}_{\overline{b}}(a) = \{0_2\}$ for a close to $\bar{a}$. As a consequence,

$$\operatorname{liplsc} \mathcal{F}_{\overline{b}}(\overline{a}, \bar{x}) = \operatorname{liplsc}^* \mathcal{F}_{\overline{b}}(\overline{a}, \bar{x}) = \operatorname{lip} \mathcal{F}_{\overline{b}}(\overline{a}, \bar{x}) = 0.$$

In the previous example (as well as in Example 1), the fact that $\overline{x} = 0_n$ was essential. Indeed, Theorem 6 below shows that for $\overline{x} \neq 0_n$ we still have equality among the three moduli and the finiteness of them is characterized by SSCQ. As a preparatory step and consistent with our previous analysis, we consider first a single inequality, $\mathcal{L}_{\overline{b}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, defined as

$$\mathcal{L}_{\overline{b}}(a) := \left\{ x \in \mathbb{R}^n \,|\, a'x \leq \overline{b} \right\}.$$

Then, for any $\overline{u} \neq 0_n$ and $\overline{x} \in \mathbb{R}^n$, we infer from the Ascoli formula that

$$\text{liplsc}\,\mathcal{L}_{\overline{u}'\overline{x}}\left(\overline{u},\overline{x}\right) = \limsup_{a\to\overline{u}, a\neq\overline{u}} \frac{d\left(\overline{x},\mathcal{L}_{\overline{u}'\overline{x}}(a)\right)}{d\left(a,\overline{u}\right)} = \limsup_{a\to\overline{u}, a\neq\overline{u}} \frac{1}{\|a\|_*}\frac{\left[\left(a-\overline{u}\right)'\overline{x}\right]_+}{\|a-\overline{u}\|_*} = \frac{\|\overline{x}\|}{\|\overline{u}\|_*}. \tag{27}$$

Theorem 6. *Assume that $\{\overline{a}_t, t\in T\}$ is bounded and let $0_n \neq \overline{x} \in \mathcal{F}\left(\overline{\sigma}\right)$. Then,*

$$\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x}) = \text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},\overline{x}) = \text{lip}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x}) = \frac{\|\overline{x}\|}{d_*\left(0_n, C_{\overline{x}}\right)},$$

where all quantities reduce to

$$\frac{\|\overline{x}\|}{d_*\left(0_n, \text{conv}\,\left\{\overline{a}_t, t\in T_{\overline{\sigma}}(\overline{x})\right\}\right)},$$

if T even happens to be a finite set.

Proof. We adapt the beginning of the proof of Theorem 5, where we just replace the mapping $\mathcal{F}_{\overline{a}}$ (considered around the point $(\overline{b},\overline{x})$) by the mapping $\mathcal{F}_{\overline{b}}$ (considered around the point $(\bar{a},\overline{x})$). This time, exploiting the relations (i) and (iii) of Lemma 3, we arrive by an analogous reasoning at the relation

$$\text{lip}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x}) \leq \frac{\|\overline{x}\|}{d_*\left(0_n, C_{\overline{x}}\right)}.$$

With the same argument as in the proof of Theorem 4, it will be sufficient to lead the relation

$$\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x}) < \frac{\|\overline{x}\|}{d_*\left(0_n, C_{\overline{x}}\right)}, \tag{28}$$

to a contradiction. As in the proofs before, assume first that $0_n \in C_{\overline{x}}$. Then, $0_{n+1} \in \text{clconv}\left(\overline{\sigma}\right)$ by (18) and we can write

$$0_{n+1} = \lim_r \sum_{t\in T} \lambda_t^r \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix},$$

for some $\{\lambda^r\}_{r\in\mathbb{N}} \subset \mathbb{R}_+^{(T)}$ with $\sum_{t\in T}\lambda_t^r = 1$ for all $r \in \mathbb{N}$. Choose $w \in \mathbb{R}^n$ with $\|w\|_* = 1$ and $w'\overline{x} = \|\overline{x}\|$. For any $\varepsilon > 0$ consider $a^\varepsilon \in \left(\mathbb{R}^n\right)^T$ given by

$$a_t^\varepsilon := \overline{a}_t + \varepsilon w \text{ for all } t \in T\ .$$

Therefore,

$$\lim_r \sum_{t\in T} \lambda_t^r \begin{pmatrix} a_t^\varepsilon \\ \overline{b}_t \end{pmatrix} = \begin{pmatrix} \varepsilon w \\ 0 \end{pmatrix}.$$

For each $\varepsilon > 0$ we either have that $\mathcal{F}_{\overline{b}}\left(a^\varepsilon\right) = \emptyset$ or (consistent case) that Lemma 1 allows us to derive the inequality $\varepsilon w'x \leq 0$ from the inequality system $a_t^{\varepsilon\prime} x \leq \overline{b}_t^r \quad (t \in T)$. In any case we get the inclusion

$$\mathcal{F}_{\overline{b}}\left(a^\varepsilon\right) \subseteq \left\{x \in \mathbb{R}^n \mid \varepsilon w'x \leq 0\right\},$$

for all $\varepsilon > 0$. Now, according to the Ascoli formula,

$$d\left(\overline{x}, \mathcal{F}_{\overline{b}}\left(a^{\varepsilon}\right)\right) \geq \frac{\varepsilon w' \overline{x}}{\varepsilon \left\|w\right\|_{*}} = \left\|\overline{x}\right\| = \frac{\left\|\overline{x}\right\|}{\varepsilon} d\left(\overline{a}, a^{\varepsilon}\right) \ .$$

Letting $\varepsilon \downarrow 0$ and recalling that $\overline{x} \neq 0_n$, we see that $\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a}, \overline{x}) = \infty$ contradicting (28).

If, in contrast, $0_n \notin C_{\overline{x}}$, then choose $0_n \neq \overline{u} \in C_{\overline{x}}$ such that $d_*\left(0_n, C_{\overline{x}}\right) = \left\|\overline{u}\right\|_*$. According to (28) and (27), we may also find some $\alpha > 0$ such that

$$\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a}, \overline{x}) < \alpha < \frac{\left\|\overline{x}\right\|}{\left\|\overline{u}\right\|_*} = \text{liplsc}\mathcal{L}_{\overline{u}'\overline{x}}\left(\overline{u}, \overline{x}\right). \tag{29}$$

Consider a sequence $\{u^r\}$ converging to $\overline{u}$ such that

$$d\left(\overline{x}, \mathcal{L}_{\overline{u}'\overline{x}}\left(u^r\right)\right) > \alpha \|u^r - \overline{u}\|_*, \ r = 1, 2, ...$$

Then, repeating the arguments in the end of the proof of Theorem 5, but applied to the sequence $\left(a^r, \overline{b}\right)$, where $a_t^r := \overline{a}_t + u^r - \overline{u}$, we derive the inclusion $\mathcal{F}_{\overline{b}}\left(a^r\right) \subseteq \mathcal{L}_{\overline{u}'\overline{x}}\left(u^r\right)$ for r sufficiently large and finally the contradiction

$$d\left(\overline{x}, \mathcal{F}_{\overline{b}}\left(a^r\right)\right) \geq d\left(\overline{x}, \mathcal{L}_{\overline{u}'\overline{x}}\left(u^r\right)\right) > \alpha \|u^r - \overline{u}\|_* = \alpha d\left(a^r, a\right),$$

with the first inequality of (29). □

The case of $\bar{x} = 0_n \in \mathcal{F}\left(\overline{\sigma}\right)$ is more delicate according to whether SSCQ is satisfied or not and if not, then whether 0_n is the unique element of $\mathcal{F}\left(\overline{\sigma}\right)$ or not.

Theorem 7. *Assume that* $\{\overline{a}_t, t \in T\}$ *is bounded and* $0_n \in \mathcal{F}\left(\overline{\sigma}\right)$. *Then,*

(*i*) $\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = 0$,

(*ii*) *If SSCQ is satisfied at* $\overline{\sigma}$, *then* $\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = \text{lip}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = 0$.

(*iii*) *If SSCQ is violated at* $\overline{\sigma}$, *then*

- *a) If* $\mathcal{F}(\overline{\sigma}) \neq \{0_n\}$, *then* $\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = \text{lip}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = \infty$
- *b) If* $\mathcal{F}(\overline{\sigma}) = \{0_n\}$, *then* $\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = 0$
- *c) If* $\mathcal{F}(\overline{\sigma}) = \{0_n\}$ *and* T *is finite, then* $\text{lip}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = 0$.

Proof. (i) The fact that $0_n \in \mathcal{F}_{\overline{b}}\left(\overline{a}\right)$ implies that $\overline{b}_t \geq 0$ for all $t \in T$, hence

$$0_n \in \mathcal{F}_{\overline{b}}\left(a\right) \quad \forall a \in \left(\mathbb{R}^n\right)^T. \tag{30}$$

The assertion follows from (13).

(ii) From (14),

$$\text{lip}\mathcal{F}_{\overline{b}}(\overline{a}, \overline{x}) = \limsup_{x \to 0_n, a \to \overline{a}, a \notin \mathcal{F}_{\overline{b}}^{-1}(x)} \frac{d\left(x, \mathcal{F}_{\overline{b}}(a)\right)}{d\left(a, \mathcal{F}_{\overline{b}}^{-1}(x)\right)}.$$

Observe, that the relation $a \notin \mathcal{F}_{\overline{b}}^{-1}(x)$ implies $x \neq 0$ by (30). Exploiting once more the relations (i) and (iii) of Lemma 3 and (14), we may continue as

$$\begin{aligned}\operatorname{lip}\mathcal{F}_{\overline{b}}(\overline{a},0_n) &= \limsup_{x\to 0_n, a\to\overline{a}, a\notin\mathcal{F}_{\overline{b}}^{-1}(x)} \frac{\|x\|\, d\left(x,\mathcal{F}(a,\overline{b})\right)}{(1+\|x\|)\ d\left((a,\overline{b}),\mathcal{F}^{-1}(x)\right)} \\ &\leq \limsup_{x\to 0_n, \sigma\to\overline{\sigma}, \sigma\notin\mathcal{F}^{-1}(x)} \frac{\|x\|\, d\left(x,\mathcal{F}(\sigma)\right)}{(1+\|x\|)\ d\left(\sigma,\mathcal{F}^{-1}(x)\right)} \\ &= \limsup_{x\to 0_n} \frac{\|x\|}{1+\|x\|}\cdot \operatorname{lip}\mathcal{F}(\overline{\sigma},0_n) = 0,\end{aligned}$$

where we used the fact that $\operatorname{lip}\mathcal{F}(\overline{\sigma},0_n)<\infty$ under SSCQ (see equivalences (i) and (iii) in Theorem 2 as well as (9)). The assertion follows from (10).

(iii) Assume that SSCQ is violated at $\overline{\sigma}$, whence $0_{n+1}\in \operatorname{cl}\operatorname{conv}(\overline{\sigma})$ by Theorem 2. Turning to subcase (a), choose any $\widehat{x}\in\mathcal{F}_{\overline{b}}(\overline{a})\setminus\{0_n\}$ and any $\varepsilon>0$. Define

$$\widehat{a}_t^{\varepsilon} := \overline{a}_t + \varepsilon^2 w \text{ for all } t\in T\ ,$$

where $\|w\|_* = 1$ and $w'\widehat{x}=\|\widehat{x}\|$. Then, analogously to case '$0_n\in C_{\overline{x}}$' in Theorem 6 we conclude that

$$\mathcal{F}_{\overline{b}}\left(\widehat{a}^{\varepsilon}\right)\subseteq\left\{x\in\mathbb{R}^n : \varepsilon^2 w'x\leq 0\right\},$$

for all $\varepsilon>0$ and by Ascoli formula

$$d\left(\varepsilon\widehat{x},\mathcal{F}_{\overline{b}}\left(\widehat{a}^{\varepsilon}\right)\right)\geq\frac{\varepsilon^3 w'\widehat{x}}{\varepsilon^2}=\varepsilon\|\widehat{x}\|=\frac{\|\widehat{x}\|}{\varepsilon}d\left(\overline{a},\widehat{a}^{\varepsilon}\right).$$

Note that, as $\varepsilon\downarrow 0$, $\varepsilon\widehat{x}$ becomes arbitrarily close to $\overline{x}=0_n$ and $\widehat{a}^{\varepsilon}\to\overline{a}$, so that we have shown that $\operatorname{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},0_n)=\infty$ which also implies by (10) that $\operatorname{lip}\mathcal{F}_{\overline{b}}(\overline{a},0_n)=\infty$. In subcase (b), the additional assumption $\mathcal{F}(\overline{\sigma})=\{0_n\}$ shows via (30) that

$$d(x,\mathcal{F}_{\overline{b}}(a))\leq\|x\|=0\quad\forall x\in\mathcal{F}_{\overline{b}}(\overline{a})=\mathcal{F}(\overline{\sigma})=\{0_n\}.$$

Now, $\operatorname{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},0_n)=0$ follows from (12). Finally, in subcase (c), we additionally assume that T is finite and show that

$$0_n\in\operatorname{intconv}\left\{\overline{a}_t | t\in T_{\overline{\sigma}}(0_n)\right\}. \tag{31}$$

Indeed, otherwise the separation theorem would yield the existence of some $u\neq 0_n$ such that $u'z\leq 0$ for all $z\in\operatorname{conv}\left\{\overline{a}_t,\ t\in T_{\overline{\sigma}}(0_n)\right\}$. Define $\tilde{u}:=\tau u\neq 0_n$ where

$$\tau:=\min_{t\in T}\left\{\frac{\overline{b}_t}{\|u\|\,\|\bar{a}_t\|_*}\,\middle|\,\overline{b}_t>0, \|\bar{a}_t\|_*\neq 0\right\}>0.$$

(recall that T is finite). Then, taking into account that $\overline{b}_t\geq 0$ for all $t\in T$ (see proof of case (i)), we arrive at

$$\tilde{u}'\bar{a}_t\leq\begin{cases}0=\overline{b}_t & \text{if } t\in T_{\overline{\sigma}}(0_n)\quad\left(\overline{b}_t=0\right),\\ \tau\|u\|\,\|\bar{a}_t\|_*\leq\overline{b}_t & \text{if } t\in T\backslash T_{\overline{\sigma}}(0_n)\quad\left(\overline{b}_t>0\right),\end{cases}$$

whence the contradiction $\tilde{u}\in\mathcal{F}_{\overline{b}}(\overline{a})\backslash\{0_n\}$ with $\mathcal{F}(\overline{\sigma})=\{0_n\}$. Now, (31) implies (see, e.g., [10, Exercise 6.12]), that for a close enough to $\overline{a}$

$$0_n\in\operatorname{intconv}\left\{a_t | t\in T_{\overline{\sigma}}(0_n)\right\}. \tag{32}$$

Assume that there exists some $0_n \neq u \in \mathcal{F}_{\overline{b}}(a)$ for some a sufficiently close to $\overline{a}$ as to satisfy (32). Then, $0 \neq \varepsilon u \in \text{conv}\ \{a_t | t \in T_{\overline{\sigma}}(0_n)\}$ for some $\varepsilon > 0$. Hence, there exist multipliers $\lambda_t \geq 0$ for $t \in T_{\overline{\sigma}}(0_n)$ such that

$$\varepsilon u = \sum_{t \in T_{\overline{\sigma}}(0_n)} \lambda_t a_t,$$

whence the contradiction

$$0 < \varepsilon u' u = \sum_{t \in T_{\overline{\sigma}}(0_n)} \lambda_t a_t' u \leq \sum_{t \in T_{\overline{\sigma}}(0_n)} \lambda_t \overline{b}_t = 0,$$

where the last equality follows from the fact that $\overline{b}_t = 0$ for $t \in T_{\overline{\sigma}}(0_n)$. Consequently, by (30),

$$\mathcal{F}_{\overline{b}}(a) = \{0_n\}, \text{ for } a \text{ close enough to } \overline{a},$$

yielding $\text{lip}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n) = 0$. □

A comparison of all cases in Theorems 6 and 7 shows the following equivalence for the partial mapping $\mathcal{F}_{\overline{b}}$, which—unlike the mappings $\mathcal{F}_{\overline{a}}$ and $\mathcal{F}$—cannot be extended to the Lipschitz-lsc property (see Example 1):

Corollary 1. *If T is finite, then $\mathcal{F}_{\overline{b}}$ has the Aubin property at $(\overline{a}, \overline{x}) \in \text{gph}\mathcal{F}_{\overline{b}}$ if and only if it is Lipschitz-lsc* at the same point. Moreover, the corresponding moduli coincide.*

However, Theorems 6 and 7 also show that full equivalence continues to hold under SSCQ and the boundedness of LHS coefficients:

Corollary 2. *If SSCQ is satisfied at $\overline{\sigma}$ and $\{\overline{a}_t,\ t \in T\}$ is bounded, then $\mathcal{F}_{\overline{b}}$ has the Aubin property at $(\overline{a}, \overline{x}) \in \text{gph}\mathcal{F}_{\overline{b}}$ if and only if it is Lipschitz-lsc*, which happens if and only if it is Lipschitz-lsc at the same point. Moreover, the corresponding moduli coincide.*

The computation of $\text{lip}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n)$ in the case when $\mathcal{F}(\overline{\sigma}) = \{0_n\}$, SSCQ fails, and T is infinite remains as an open problem. Indeed, the following example shows that $\text{lip}\mathcal{F}_{\overline{b}}(\overline{a}, 0_n)$ can be either finite or infinite in this particular case.

Example 4. For any given $p > 1$ let us consider the following nominal linear inequality system, in $\mathbb{R}$ (endowed with the norm of the absolute value),

$$\{tx \leq |t|^p,\ t \in [-1, 1]\}$$

(in other words, $\overline{a}_t = t$ and $\overline{b}_t = |t|^p$ for all $t \in [-1, 1]$), whose unique feasible solution is $\overline{x} = 0$. For *any* $0 < \varepsilon \leq 1 - 1/p$ let us consider the perturbed system

$$\{(t - \varepsilon)\, x \leq |t|^p,\ t \in [-1, 1]\},$$

whose feasible set is the interval $\left[0, p^p (p-1)^{1-p} \varepsilon^{p-1}\right]$. Indeed, it can be checked that, for each $a \in \mathbb{R}^{\mathbb{R}}$ we have

$$\|a - \overline{a}\| \leq \varepsilon \Rightarrow \mathcal{F}_{\overline{b}}(a) \subset \left[-p^p (p-1)^{1-p} \varepsilon^{p-1}, p^p (p-1)^{1-p} \varepsilon^{p-1}\right].$$

Accordingly,

$$\operatorname{lip}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x}) = \begin{cases} \infty & \textit{if } 1 < p < 2, \\ 4 & \textit{if } p = 2, \\ 0 & \textit{if } p > 2. \end{cases}$$

6. The metric-regularity counterpart of the Lipschitz-lsc modulus

In this section, we add an observation concerning the so-called 'metric-regularity counterpart' of Lipschitz-lsc in the context of multifunctions considered in this paper. By this, we mean the expression

$$\limsup_{y\to\bar{y}, y\notin\mathcal{G}^{-1}(\bar{x})} \frac{d(\bar{x},\mathcal{G}(y))}{d(y,\mathcal{G}^{-1}(\bar{x}))}, \tag{33}$$

for a general multifunction $\mathcal{G}: Y \rightrightarrows X$ around some point $(\bar{y},\bar{x})$ of its graph. Apart from fixing $\bar{x}$ in this expression, it looks similar to the modulus of metric regularity (14) of $\mathcal{G}^{-1}$ at $(\bar{x},\bar{y})$ which equals the Lipschitz modulus of $\mathcal{G}$ at $(\bar{y},\bar{x})$. Since $\bar{x}$ is also fixed in the formula for the Lipschitz-lsc modulus (13), the natural question arises if (33) is equivalent with (13). The following relations are obvious from (13) and (14):

$$\operatorname{liplsc}\mathcal{G}(\overline{y},\overline{x}) \le \limsup_{y\to\bar{y}, y\notin\mathcal{G}^{-1}(\bar{x})} \frac{d(\bar{x},\mathcal{G}(y))}{d(y,\mathcal{G}^{-1}(\bar{x}))} \le \operatorname{lip}\mathcal{G}(\overline{y},\overline{x}). \tag{34}$$

In general, both inequalities can be strict. Example 1 shows this for the second inequality. Since therein $y \in \mathcal{G}^{-1}(0)$ for any $y \in \mathbb{R}$, the superior limit of non-negative numbers in (33) is taken over the empty set, hence formally equal to zero:

$$0 = \limsup_{y\to 0, y\notin\mathcal{G}^{-1}(0)} \frac{d(0,\mathcal{G}(y))}{d(y,\mathcal{G}^{-1}(0))} < \operatorname{lip}\mathcal{G}(0,0) = \infty,$$

while the Lipschitz modulus equals infinity because the Aubin property fails to hold for $\mathcal{G}$ at $(0,0)$. The next example shows that the first inequality in (34) can be strict too:

Example 5. Consider the (single-valued) mapping $\mathcal{G}(y) := \{f(y)\}$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by:

$$f(y) := \begin{cases} y\sin\frac{1}{y} & \textit{if} \quad y \neq 0 \\ 0 & \textit{if} \quad y = 0\ . \end{cases}$$

Here, we have that

$$\limsup_{y\to 0} \frac{d(0,\mathcal{G}(y))}{d(y,\mathcal{G}^{-1}(0))} = \infty, \tag{35}$$

while

$$\operatorname{liplsc}\mathcal{G}(0,0) = \operatorname{liplsc}^{*}\mathcal{G}(0,0) = 1. \tag{36}$$

In order to prove (35), we observe that

$$\mathcal{G}^{-1}(0) = \{0\} \cup \left\{ \frac{1}{k\pi} \;\middle|\; k \in \mathbb{Z}\setminus\{0\} \right\}\ .$$

For $k \in \mathbb{N}$, consider the sequence $\{y_k\}$ defined as:

$$y_k := \frac{1}{2}\left(\frac{1}{2k\pi} + \frac{1}{(2k+1)\pi}\right) = \frac{4k+1}{4k(2k+1)\pi} .$$

Then,

$$d\left(y_k, \mathcal{G}^{-1}(0)\right) = \frac{1}{2}\left(\frac{1}{2k\pi} - \frac{1}{(2k+1)\pi}\right) = \frac{1}{4k(2k+1)\pi} .$$

Hence,

$$\frac{d\left(0, \mathcal{G}(y_k)\right)}{d\left(y_k, \mathcal{G}^{-1}(0)\right)} = \frac{\frac{4k+1}{4k(2k+1)\pi}\sin\frac{4k(2k+1)\pi}{4k+1}}{\frac{1}{4k(2k+1)\pi}} = (4k+1)\sin\frac{2k\pi}{4k+1},$$

which tends to infinite when $k \to \infty$. Therefore,

$$\limsup_{y\to 0}\frac{d\left(0, \mathcal{G}(y)\right)}{d\left(y, \mathcal{G}^{-1}(0)\right)} = \infty.$$

The identity (36) follows from the fact that for continuous functions Lipschitz-lsc and Lipschitz-lsc* are equivalent properties and their moduli at $(0, f(0))$ coincide with the following expression:

$$\limsup_{y\to 0}\frac{d\left(f(0), f(y)\right)}{d\left(0, y\right)} = \limsup_{y\to 0}\frac{\left|y\sin\frac{1}{y}\right|}{|y|} = 1.$$

In the context of our linear constraint mappings (16), however, the metric-regularity counterpart of Lipschitz-lsc turns out to be identical with the moduli of Lipschitz-lsc and Lipschitz-lsc* (where in the case of the partial mapping $\mathcal{F}_{\overline{b}}$ one has to impose an additional assumption):

Proposition 1. *In (16), assume that $\{\bar{a}_t \mid t \in T\}$ is bounded and consider any $(\overline{\sigma}, \overline{x}) \in \operatorname{gph}\mathcal{F}$. Then,*

$$\operatorname{liplsc}\mathcal{F}\left(\overline{\sigma}, \overline{x}\right) = \operatorname{liplsc}^*\mathcal{F}\left(\overline{\sigma}, \overline{x}\right) = \limsup_{\sigma\to\bar{\sigma}, \sigma\notin\mathcal{F}^{-1}(\bar{x})}\frac{d\left(\bar{x}, \mathcal{F}(\sigma)\right)}{d\left(\sigma, \mathcal{F}^{-1}(\bar{x})\right)};$$

$$\operatorname{liplsc}\mathcal{F}_{\bar{a}}\left(\overline{b}, \overline{x}\right) = \operatorname{liplsc}^*\mathcal{F}_{\bar{a}}\left(\overline{b}, \overline{x}\right) = \limsup_{b\to\bar{b}, b\notin\mathcal{F}_{\bar{a}}^{-1}(\bar{x})}\frac{d\left(\bar{x}, \mathcal{F}_{\bar{a}}\left(b\right)\right)}{d\left(\sigma, \mathcal{F}_{\bar{a}}^{-1}(\bar{x})\right)}.$$

Moreover, if $\overline{x} \neq 0_n$, then

$$\operatorname{liplsc}\mathcal{F}_{\bar{b}}\left(\overline{a}, \overline{x}\right) = \operatorname{liplsc}^*\mathcal{F}_{\bar{b}}\left(\overline{a}, \overline{x}\right) = \limsup_{a\to\bar{a}, a\notin\mathcal{F}_{\bar{b}}^{-1}(\bar{x})}\frac{d\left(\bar{x}, \mathcal{F}_{\bar{b}}\left(a\right)\right)}{d\left(\sigma, \mathcal{F}_{\bar{b}}^{-1}(\bar{x})\right)}.$$

Proof. The proof follows immediately from the relations (10) and (34) upon taking into account Theorems 4, 5 and 6. □

7. Conclusions

By combining the results of the preceding sections we can construct the following table of conclusions. Recall that we are working under the assumption that $\{\overline{a}_t, t \in T\}$ is bounded. Recall also that $C_{\overline{x}}$ was defined in (18).

Full perturbations		RHS perturbations		LHS perturbations
$\text{liplsc}\mathcal{F}(\overline{\sigma},\overline{x})$	$=$	$\text{liplsc}\mathcal{F}_{\overline{a}}(\overline{b},\overline{x})$	$+$	$\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$
$\Vert$		$\Vert$		$\Vert$
$\dfrac{\Vert\overline{x}\Vert+1}{d_*\left(0_n, C_{\overline{x}}\right)}$	$=$	$\dfrac{1}{d_*\left(0_n, C_{\overline{x}}\right)}$	$+$	$\dfrac{\Vert\overline{x}\Vert}{d_*\left(0_n, C_{\overline{x}}\right)}$
$\Vert$		$\Vert$		$\Vert^{(*)}$
$\text{liplsc}^*\mathcal{F}(\overline{\sigma},\overline{x})$	$=$	$\text{liplsc}^*\mathcal{F}_{\overline{a}}(\overline{b},\overline{x})$	$+$	$\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$
$\Vert$		$\Vert$		$\Vert^{(**)}$
$\text{lip}\mathcal{F}(\overline{\sigma},\overline{x})$	$=$	$\text{lip}\mathcal{F}_{\overline{a}}(\overline{b},\overline{x})$	$+$	$\text{lip}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$

Observe that, in contrast to Theorem 1, Corollaries 1 and 2 not only state equivalences of properties but in addition the equality of the associated moduli which can be calculated by means of explicit formulae. With respect to the previous table, let us point out the following:

- Looking at the different rows of equalities, we conclude that for any of the three moduli, $\text{liplsc}\mathcal{F}(\overline{\sigma},\overline{x})$, $\text{liplsc}^*\mathcal{F}(\overline{\sigma},\overline{x})$, and $\text{lip}\mathcal{F}(\overline{\sigma},\overline{x})$, its value can always be decomposed as the sum of the corresponding moduli under RHS and LHS perturbations.
- The first two columns of equalities have been established in Theorems 4 and 5, devoted to the frameworks of full and RHS perturbations, respectively.
- Theorems 6 and 7 gather the results about the third column of equalities, devoted to LHS perturbations. $^{(*)}$: this equality is held with the only exception when $\overline{x}=0_n$, $\mathcal{F}(\overline{\sigma})\neq\{0_n\}$ and SSCQ fails (in which case, $\text{liplsc}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})=0$, while $\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})=\infty$). $^{(**)}$: this equality holds with the only exception when $\mathcal{F}(\overline{\sigma})=\{0_n\}$ (hence $\overline{x}=0_n$), SSCQ fails, and T is infinite; in this case $\text{liplsc}^*\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})=0$, while $\text{lip}\mathcal{F}_{\overline{b}}(\overline{a},\overline{x})$ is undetermined.

Remark 1. The case $\mathcal{F}_{\overline{b}}(\overline{a})=\{0_n\}$ in the parametric setting of LHS perturbations may be viewed as a particular case of $\widetilde{\mathcal{F}}(\overline{a})=\{\overline{x}\}$, where the nominal data $\left(\overline{a},\overline{b}\right)\in(\mathbb{R}^n)^T\times\mathbb{R}^T$ and $\overline{x}\in\mathbb{R}^n$ are fixed and $\widetilde{\mathcal{F}}:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is given by

$$\widetilde{\mathcal{F}}(a) := \left\{x\in\mathbb{R}^n : a_t'x \leq \overline{b}_t + (a_t-\overline{a}_t)'\,\overline{x} \text{ for all } t\in T\right\};$$

i.e., both sides of the linear inequality system are perturbed in such a way that the feasibility of $\overline{x}$ is preserved. In view of Example 4, the study of the Lipschitz behavior of mapping $\widetilde{\mathcal{F}}$ at $(\overline{a},\overline{x})$ perhaps requires a certain second-order analysis which is out of the scope of this paper; it could constitute a subject for further research.

References

[1] J.-P. Aubin, Lipschitz behavior of solutions to convex minimization problems, Math. Oper. Res. 9 (1984) 87–111.

[2] M. Benko, H. Gfrerer, J.V. Outrata, Calculus for directional limiting normal cones and subdifferentials, Set-Valued Var. Anal. 27 (2019) 713–745.

[3] M.J. Cánovas, A.L. Dontchev, M.A. López, J. Parra, Metric regularity of semi-infinite constraint systems, Math. Program., Ser. A 104 (2005) 329–346.

[4] M.J. Cánovas, M.A. López, J. Parra, F.J. Toledo, Distance to ill-posedness and the consistency value of linear semi-infinite inequality systems, Math. Program. 103A (2005) 95–126.

[5] M.J. Cánovas, F.J. Gómez-Senent, J. Parra, Regularity modulus of arbitrarily perturbed linear inequality systems, J. Math. Anal. Appl. 343 (2008) 315–327.

[6] M.J. Cánovas, M.A. López, J. Parra, F.J. Toledo, Calmness of the feasible set mapping for linear inequality systems, Set-Valued Var. Anal. 22 (2014) 375–389.

[7] A. Daniilidis, M.A. Goberna, M.A. López, R. Lucchetti, Stability in linear optimization under perturbations of the left-hand side coefficients, Set-Valued Var. Anal. 23 (2015) 737–758.
[8] A.L. Dontchev, R.T. Rockafellar, Implicit Functions and Solution Mappings: A View from Variational Analysis, Springer, New York, 2009.
[9] H. Gfrerer, J.V. Outrata, On Lipschitzian properties of implicit multifunctions, SIAM J. Optim. 26 (2016) 2160–2189.
[10] M.A. Goberna, M.A. López, Linear Semi-Infinite Optimization, John Wiley & Sons, Chichester (UK), 1998.
[11] R. Henrion, A. Jourani, J. Outrata, On the calmness of a class of multifunctions, SIAM J. Optim. 13 (2002) 603–618.
[12] A.J. Hoffman, On approximate solutions of systems of linear inequalities, J. Res. Natl. Bur. Stand. 49 (1952) 263–265.
[13] A. Jourani, Hoffman's error bound, local controllability and sensitivity analysis, SIAM J. Control Optim. 38 (2000) 947–970.
[14] D. Klatte, A Note on Quantitative Stability Results in Nonlinear Programming, Seminarbericht, vol. 90, Der Sektion Mathematik der Humboldt-Universität zu Berlin, 1987, pp. 77–86.
[15] D. Klatte, B. Kummer, Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications, Nonconvex Optim. Appl., vol. 60, Kluwer Academic, Dordrecht, The Netherlands, 2002.
[16] D. Klatte, B. Kummer, Optimization methods and stability of inclusions in Banach spaces, Math. Program., Ser. B 117 (2009) 305–330.
[17] D. Klatte, B. Kummer, On calmness of the argmin mapping in parametric optimization problems, J. Optim. Theory Appl. 165 (2015) 708–719.
[18] D. Klatte, G. Thiere, Error bounds for solutions of linear equations and inequalities, ZOR Math. Meth. Oper. Res. 41 (1995) 191–214.
[19] W. Li, The sharp Lipschitz constants for feasible and optimal solutions of a perturbed linear program, Linear Algebra Appl. 187 (1993) 15–40.
[20] O.L. Mangasarian, T.H. Shiao, Lipschitz continuity of solutions of linear inequalities, programs and complementarity problems, SIAM J. Control Optim. 25 (1987) 583–595.
[21] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, I: Basic Theory, Springer, Berlin, 2006.
[22] S.M. Robinson, Bounds for error in the solution set of a perturbed linear program, Linear Algebra Appl. 6 (1973) 69–81.
[23] R.T. Rockafellar, R.J-B. Wets, Variational Analysis, Springer, Berlin, 1998.
[24] R. Römisch, R. Schultz, Distribution sensitivity in stochastic programming, Math. Program. 50 (1991) 197–226.
[25] A. Uderzo, On Lipschitz semicontinuity properties of variational systems with application to parametric optimization, J. Optim. Theory Appl. 162 (2014) 47–78.
[26] A. Uderzo, On a quantitative semicontinuity property of variational systems with applications to perturbed quasidifferentiable optimization, in: V. Demyanov, P.M. Pardalos, M. Batsyn (Eds.), Constructive Nonsmooth Analysis and Related Topics, Springer, Berlin, 2014, pp. 115–136.