JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 72, No. 1, JANUARY 1992

TECHNICAL NOTE

On Constraint Qualifications[1]

R. HENRION[2]

Communicated by O. L. Mangasarian

Abstract. The linear independence constraint qualification (LICQ) and the weaker Mangasarian–Fromovitz constraint qualification (MFCQ) are well-known concepts in nonlinear optimization. A theorem is proved suggesting that the set of feasible points for which MFCQ essentially differs from LICQ is small in a specified sense. As an auxiliary result, it is shown that, under MFCQ, the constraint set (even in semi-infinite optimization) is locally representable in epigraph form.

Key Words. Nonlinear optimization, constraint qualifications, representation of constraint sets, semi-infinite optimization.

1. Introduction

Constraint qualifications are of extraordinary interest in optimization problems including (in)equality restrictions. The linear independence constraint qualification (LICQ) and the Mangasarian–Fromovitz constraint qualification (MFCQ) are among the most important conditions in nonlinear optimization. Both are essential for the implementation of solution algorithms [e.g., MFCQ in homotopy methods (Ref. 1)], for stability investigations (Refs. 2–4), or for a structural analysis in parametric optimization (Refs. 5, 6). In refs. 4 and 6, LICQ and MFCQ are related to the geometrical properties of the constraint set.

It is well known that MFCQ is weaker than LICQ. In this paper, however, it is confirmed that the difference between both conditions is small in the following sense. Let $M^0 \subseteq M$ consist of those points of a constraint set M which have at least one binding inequality restriction. Evidently,

[1]The author wishes to thank Professor H. T. Jongen for valuable advice.

[2]Assistant, Institute of Applied Mathematics, Humboldt University of Berlin, Berlin, Germany.

MFCQ and LICQ are equivalent in $M\backslash M^0$; therefore, only M^0 is of interest. Now, let MFCQ be fulfilled for M^0, and consider the subset $A \subseteq M^0$ having the property that, at all points of A, the constraint set M may be locally described by a representation satisfying LICQ. An easy example shows that, in general, $A \neq M^0$; nevertheless, it turns out that A is open and dense in the relative topology of M^0. The corresponding theorem requires essentially a local description of the constraint set in epigraph form. Therefore, first a lemma is proved showing that this is always possible under MFCQ. Since the last fact is not only true in finite optimization, but even in semi-infinite optimization, we prove a bit more than needed in the theorem.

2. Results

Let

$$M = \{z \in \mathbb{R}^n \mid h_j(z) = 0,\ j = 1, \ldots, p,\quad \text{and } g_i(z) \geq 0,\ i = 1, \ldots, s\}, \tag{1}$$

with $h_j, g_i \in C^1(\mathbb{R}^n, \mathbb{R})$, be an ordinary constraint set in finite-dimensional optimization. For $z^0 \in M$, denote by

$$I(z^0) := \{i \in \{1, \ldots, s\} \mid g_i(z^0) = 0\}$$

the set of active indices.

Definition 2.1. The linear independence constraint qualification (LICQ) is said to hold at $z^0 \in M$ if the set of gradients

$$\{Dh_j(z^0)\}_{j=1,\ldots,p} \cup \{Dg_i(z^0)\}_{i \in I(z^0)}$$

is linearly independent.

Definition 2.2. The Mangasarian–Fromovitz constraint qualification (MFCQ) is said to hold at $z^0 \in M$ if:

(a) rank $\{Dh_j(z^0)\}_{j=1,\ldots,p} = p$;
(b) there exists a vector $\xi \in \mathbb{R}^n$ satisfying $Dh_j(z^0) \cdot \xi = 0$, $j = 1, \ldots, p$, and $Dg_i(z^0) \cdot \xi > 0$, $i \in I(z^0)$.

Obviously, LICQ implies MFCQ, whereas the converse is not true. In semi-infinite programming, a constraint set is defined as

$$B = \{z \in \mathbb{R}^n \mid h_j(z) = 0,\ j = 1, \ldots, p, \text{ and } g(x, z) \geq 0,\ \forall x \in K\},$$

where $K \subseteq \mathbb{R}^m$ is compact, $h_j \in C^1(\mathbb{R}^n, \mathbb{R}), j=1, \ldots, p$, and $g \in C^1(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$. In the case of a finite index set K, one obviously deals with an ordinary constraint set in finite-dimensional optimization. For $z^0 \in B$, denote the set of active indices by

$$E(z^0) = \{x \in K \mid g(x, z^0) = 0\}.$$

Evidently, $E(z^0)$ is compact as well.

Definition 2.3. See Ref. 7. The extended Mangasarian–Fromovitz constraint qualification (EMFCQ) is said to hold at $z^0 \in B$ if:

(a) rank $\{Dh_j(z^0)\}_{j=1,\ldots,p} = p$;
(b) there exists a vector $\xi \in \mathbb{R}^n$ satisfying $Dh_j(z^0) \cdot \xi = 0, j=1, \ldots, p$, and $D_z g(x, z^0) \cdot \xi > 0, \forall x \in E(z^0)$.

Both MFCQ and EMFCQ imply $p \leq n$ and, if the set of active indices is nonempty, $\xi \neq 0$. Furthermore, EMFCQ coincides with MFCQ in finite-dimensional programming if K is a finite set.

Lemma 2.1. If $0 \leq p \leq n-2$, then for each $z^0 \in B$ with $E(z^0) \neq \emptyset$, the following statements are equivalent:

(i) EMFCQ is fulfilled at z^0;
(ii) there exists a local C^1-coordinate transformation $y = \psi(z)$ mapping z^0 onto $0 \in \mathbb{R}^n$ such that, in the new coordinates, the defining (in)equalities for B locally become

$$y_1 \geq \phi(x, y_2, \ldots, y_{n-p}), \qquad \forall x \in K^*,$$

and

$$y_{n-p+1} = \cdots = y_n = 0 \qquad \text{if } p \geq 1;$$

here, K^* is a compact set with

$$E(z^0) \subseteq K^* \subseteq K \quad \text{and} \quad \phi \in C^1(K^* \times V, \mathbb{R}),$$

with V being an open neighborhood of $0 \in \mathbb{R}^{n-p-1}$.

Proof.
(i)→(ii) Let EMFCQ be fulfilled at z^0, and define a coordinate transformation ψ^a by

$$(\psi^a)_i(z) := v_i^T(z^0 - z), \qquad i=1, \ldots, n-p,$$

$$(\psi^a)_{n-p+j}(z) := h_j(z), \qquad j=1, \ldots, p,$$

where $v_i \in \mathbb{R}^n$ are vectors with

$$\text{span}\{\{v_i\}_{i=1,\dots,n-p} \cup \{Dh_j(z^0)\}_{j=1,\dots,p}\} = \mathbb{R}^n;$$

see Definition 2.3(a). Since $D\psi^a(z^0)$ is nonsingular, ψ^a represents a C^1-diffeomorphism of an open neighborhood $\mathcal{O}_1(z^0)$ onto a neighborhood of $0 \in \mathbb{R}^n$. Put

$$\xi^* := D\psi^a(z^0) \cdot \xi / |D\psi^a(z^0) \cdot \xi|,$$

with ξ from Definition 2.3(b). Note that $\xi \neq 0$. By the same definition, it follows that $\xi^{*T} = (\xi^{\#T}, 0^T)$, where $\xi^{\#} \in \mathbb{R}^{n-p}$ and $0 \in \mathbb{R}^p$. Evidently, $\xi^{\#T}\xi^{\#} = 1$; hence, one may choose an $(n-p) \times (n-p-1)$ matrix W satisfying

$$(\xi^{\#} \,|\, W)^T(\xi^{\#} \,|\, W) = I_{n-p}.$$

Now, put

$$(\psi^b)_1(y) := \xi^{*T}y, \quad (\psi^b)_{i+1}(y) := (w_i^T \,|\, 0^T) \cdot y, \qquad i = 1, \dots, n-p-1,$$

$$(\psi^b)_j(y) := y_j, \qquad j = n-p+1, \dots, n,$$

with w_i being the ith column of W. One easily verifies that ψ^b is an invertible linear mapping with

$$[D\psi^b(0)]^T = [D\psi^b(0)]^{-1}.$$

Therefore, $\psi := \psi^b \circ \psi^a$ yields a C^1-diffeomorphism ψ: $\mathcal{O}_1(z^0) \to \mathcal{O}_2(0)$, with $\psi(z^0) = 0$ and $\mathcal{O}_2$ an open neighborhood of $0 \in \mathbb{R}^n$. Defining $g^*(x, y) := g(x, \psi^{-1}(y))$, we have $g^* \in C^1(\mathbb{R}^m \times \mathcal{O}_2(0), \mathbb{R})$ and

$$\psi(B \cap \mathcal{O}_1(z^0)) = \{y \in \mathcal{O}_2(0) \,|\, y_{n-p+1} = \cdots = y_n = 0 \text{ and } g^*(x, y) \geq 0, \ \forall x \in K\}. \tag{2}$$

Now, let $x^0 \in E(z^0)$ be an arbitrary active index; hence,

$$g^*(x^0, 0) = g(x^0, z^0) = 0. \tag{3}$$

Since the first column of

$$D\psi^{-1}(0) = [D\psi^a(z^0)]^{-1} \cdot [D\psi^b(0)]^T$$

equals $\xi / |D\psi^a(z^0) \cdot \xi|$, one obtains by Definition 2.3(b)

$$D_{y_1}g^*(x^0, 0) = D_z g(x^0, z^0) \cdot \xi / |D\psi^a(z^0) \cdot \xi| > 0. \tag{4}$$

According to the implicit function theorem applied to the equation $g^*(x, y_1, \dots, y_{n-p}, 0, \dots, 0) = 0$, (3) and (4) yield the existence of open neighborhoods U_{x^0} of x^0 and $(-\delta_{x^0}, \delta_{x^0}) \times V_{x^0} \subseteq \mathbb{R}^1 \times \mathbb{R}^{n-p-1}$ of $0 \in \mathbb{R}^{n-p}$ as well as a function $\phi_{x^0}\in C^1(U_{x^0}\times V_{x^0}, \mathbb{R})$ such that, for all $x\in U_{x^0}$ and $y\in(-\delta_{x^0}, \delta_{x^0})\times V_{x^0}\times\{0\}$, $0\in\mathbb{R}^p$, the following statements are true:

$$g^*(x, y)=0, \text{ if and only if } y_1=\phi_{x^0}(x, y_2, \ldots, y_{n-p}), \tag{5a}$$

$$D_{y_1}g^*(x, y)>0, \tag{5b}$$

$$|\phi_{x^0}(x, y_2, \ldots, y_{n-p})|<\delta_{x^0}. \tag{5c}$$

Note that (5c) holds, since $\phi_x{}^0(x^0, 0, \ldots, 0)=0$ by (3) and (5a). Considering similar neighborhoods for all $x\in E(z^0)$, one gets an open covering $\bigcup_{x\in E(z^0)} U_x$ of the compact set $E(z^0)$. Denote by $U:=\bigcup_{i=1}^k U_{x^i}$ a finite subcovering of $E(z^0)$, and define

$$(-\delta, \delta)\times V:=\bigcap_{i=1}^k \{(-\delta_{x^i}, \delta_{x^i})\times V_{x^i}\}$$

as well as a function $\phi\in C^1(U\times V, \mathbb{R})$ by

$$\phi(x, y_2, \ldots, y_{n-p}):=\phi_{x^i}(x, y_2, \ldots, y_{n-p}), \qquad x\in U_{x^i}.$$

By (5a), ϕ is correctly defined and $\phi\in C^1$. Then, for all $x\in U$ and for all $y\in(-\delta, \delta)\times V\times\{0\}$, it holds that

$$g^*(x, y)\geq 0, \qquad \text{if and only if } y_1\geq\phi(x, y_2, \ldots, y_{n-p}). \tag{6}$$

To see this, let x and y be arbitrary elements of the indicated neighborhoods with $g^*(x, y)\geq 0$. Putting $u:=\phi(x, y_2, \ldots, y_{n-p})$, we have

$$u=\phi_{x^i}(x, y_2, \ldots, y_{n-p}), \qquad \text{for some } i,$$

and by (5c),

$$(u, y_2, \ldots, y_{n-p})\in(-\delta_{x^i}, \delta_{x^i})\times V_{x^i}.$$

Assuming $y_1<u$, one gets the contradiction [see (5a) and (5b)]

$$0\leq g^*(x, y_1, y_2, \ldots, y_n)<g^*(x, u, y_2, \ldots, y_n)=0.$$

The reverse direction of (6) is proved in a similar manner. Now, let U^* be an open set with $E(z^0)\subset U^*\subset \mathrm{cl}(U^*)\subset U$ (proper inclusions), and define $\mathcal{O}_2^*(0)\subseteq\mathcal{O}_2(0)$ such that

$$g^*(x, y)>0, \qquad \forall y\in\mathcal{O}_2^*(0),\ \forall x\in K\backslash U^*. \tag{7}$$

Note that $K\backslash U^*$ is compact and $g^*(x, 0)>0$, $\forall x\in K\backslash U^*$, by the definitions of U^* and $E(z^0)$. Now, the condition $g^*(x, y)\geq 0$, $\forall x\in K$, in (2) splits into $g^*(x, y)\geq 0$, $\forall x\in K\cap \mathrm{cl}(U^*)$ [here, (6) applies] and $g^*(x, y)\geq 0$, $\forall x\in K\backslash \mathrm{cl}(U^*)$ [here, (7) applies]. Consequently combination of (2), (6), (7) yields

$$\psi(B\cap\mathcal{O}_1^*(z^0))=\{y\in\mathcal{O}_2^{**}(0)\,|\,y_{n-p+1}=\cdots=y_n=0 \text{ and }$$
$$y_1\geq\phi(x, y_2, \ldots, y_{n-p}),\ \forall x\in K^*\},$$

where

$$\mathcal{O}_2^{**}(0) := \mathcal{O}_2^*(0) \cap \{(-\delta, \delta) \times V \times \mathbb{R}^p\},$$

$$\mathcal{O}_1^*(z^0) := \psi^{-1}(\mathcal{O}_2^{**}(0)), \qquad K^* := K \cap \mathrm{cl}(U^*).$$

(ii)→(i) Since EMFCQ is invariant under the local C^1-coordinate transformations (see Ref. 7, Lemma 3.1), it is sufficient to show that the system of (in)equalities in (ii) satisfies EMFCQ at the origin of $\mathbb{R}^n$. But this is evidently true by choosing $\xi := (1, 0, \ldots, 0) \in \mathbb{R}^n$. □

The proof of Lemma 2.1 has shown that, in general, the set K^* of indices, used for the epigraph representation, properly contains the set $E(z^0)$ of active indices. However, the difference between both sets may be kept arbitrarily small. As a consequence, one may choose $K^* = E(z^0)$ in the case of a finite set K. In detail, we have the following finite version of Lemma 2.1.

Lemma 2.2. Let M be as in (1). If $0 \leq p \leq n-2$, then for each $z^0 \in M$ with $I(z^0) \neq \emptyset$, the following statements are equivalent:

(i) MFCQ is fulfilled at z^0;
(ii) there exists a local C^1-coordinate transformation $y = \psi(z)$ mapping z^0 onto $0 \in \mathbb{R}^n$ such that, in the new coordinates, the defining (in)equalities for M locally become

$$y_1 \geq \phi_i(y_2, \ldots, y_{n-p}), \qquad i \in I(z^0),$$

and

$$y_{n-p+1} = \cdots = y_n = 0, \qquad \text{for } p \geq 1,$$

where $\phi_i \in C^1(V, \mathbb{R})$, with V being an open neighborhood of $0 \in \mathbb{R}^{n-p-1}$.

Lemma 2.1 will be used as an auxiliary result in the following, but apart from this, it has an independent meaning. We mention only two direct consequences of the lemma. On the one hand, one obtains a result derived by Jongen *et al.* (Ref. 7) confirming that, under EMFCQ, a semi-infinite constraint set B represents an $(n-p)$-dimensional topological manifold with boundary. To see this, note that, (due to Lemma 2.1), in the new local coordinates, the constraint set is described by means of the inequality

$$y_1 \geq \tilde{\phi}(y_2, \ldots, y_{n-p}),$$

where

$$\tilde{\phi}(y_2, \ldots, y_{n-p}) := \max_{x \in K^*} \phi(x, y_2, \ldots, y_{n-p})$$

is Lipschitz continuous because of the compactness of K^*. Therefore, B is locally homeomorphic with a closed half-space in $\mathbb{R}^{n-p}$ (in the terminology of Ref. 8, B is epi-Lipschitzian). A second result of the corresponding finite version of Lemma 2.1 is the theorem of Gauvin (Ref. 9), which indicates that, under MFCQ, the set of Lagrange multipliers of a Kuhn–Tucker point represents a compact polyhedron. This is verified by writing the Kuhn–Tucker conditions for the finite system of inequalities

$$y_1 \geq \phi_i(y_2, \ldots, y_{n-p}), \qquad i \in I(y),$$

where the first equation (partial derivatives with respect to y_1) yields that the set of multipliers is determined by

$$\sum_{i \in I(y)} \lambda_i = D_{y_1} f(y) \quad \text{and} \quad \lambda_i \geq 0,$$

with f as objective function; see Ref. 10.

Now, we relate MFCQ to LICQ. To this aim, consider the set M in (1) and define

$$M^0 := \{z \in M \mid I(z) \neq \emptyset\}.$$

Further, let the subset $A \subseteq M^0$ consist of those z^* being supplied with a neighborhood $U(z^*)$ and functions $g_l^* \in C^1(U(z^*), \mathbb{R})$, $l = 1, \ldots, k$, such that

$$M \cap U(z^*) = \{z \in U(z^*) \mid h_j(z) = 0, j = 1, \ldots, p,$$
$$\text{and } g_l^*(z) \geq 0, l = 1, \ldots, k\}, \tag{8}$$

and LICQ holds for (8). In other words, A consists of those points in M^0 at which M may be locally described by a system of (in)equalities satisfying LICQ. Before stating the main result, we emphasize that, in general, MFCQ does not imply $A = M^0$ or in other words: reestablishing LICQ from MFCQ using a new description of the feasible set (for instance, by deleting local redundance) may be impossible. This is confirmed by the following example.

Example 2.1. Let

$$M := \{z \in \mathbb{R}^2 \mid z_2 \geq 0 \text{ and } z_2 \geq h(z_1)\},$$

where

$$h(z_1) := \begin{cases} \sin(1/z_1)\exp(-(1/z_1^2)), & z_1 \neq 0, \\ 0, & z_1 = 0. \end{cases}$$

Both constraints are even of class C^∞, and MFCQ is fulfilled at all feasible points (because of the given epigraph representation, see Lemma 2.2). The point $0 \in \mathbb{R}^2$ belongs to M^0 (both inequalities are binding), but $0 \notin A$. To see this, note that the points $(\mp 1/(k\pi), 0)^T$, $k = 1, 2, \ldots$, are corners of M i.e., two inequalities are binding with linearly independent gradients. Therefore, in each neighborhood of the origin, there is an infinite number of corners of M. If $0 \in A$, then a local redescription of M near 0 should be possible with LICQ being satisfied. This implies that, near 0, M is locally diffeomorphic with the positive orthant of $\mathbb{R}^2$; see Ref. 11. Since corners are mapped onto corners by a local diffeomorphism and since the corner of an orthant is isolated, one gets the contradiction, hence $A \neq M^0$.

The following theorem makes evident that A is at least not much smaller than M^0.

Theorem 2.1. If MFCQ holds at all points $z \in M$, then A is open and dense in the relative topology of M^0.

Proof. We show that A is dense, since the openness is clear from the definition of LICQ. First, the validity of the theorem is verified for the case $n-1 \leq p \leq n$. If $p = n$, then MFCQ implies that $I(z) = \emptyset$, $\forall z \in M$; i.e., $A = M^0 = \emptyset$. For $p = n-1$, it holds again that $A = M^0$; indeed, in this case, M^0 (if nonempty) merely consists of isolated points, at which the one-dimensional manifold $\{h_j(z) = 0\}_{j=1,\ldots,n-1}$ is transversely intersected by each of the s hypersurfaces $\{g_i(z) = 0\}$; see Definition 2.2. Therefore, near a point of M^0, all inequalities except one may be deleted to obtain another local description of M as required in (8), with $k = 1$. For one single inequality constraint, however, MFCQ and LICQ are equivalent, hence $A = M^0$. Summarizing, one may restrict the analysis to the case $0 \leq p \leq n-2$, which is required in Lemma 2.1.

It has to be shown that, in each neighborhood of an arbitrary point $z^0 \in M^0$, there is a point $z^* \in A$. We proceed by induction over the number of binding inequalities. If $\#I(z^0) = 1$, then near z^0, M is locally described by only one inequality; therefore, $z^0 \in A$ (MFCQ and LICQ equivalent). Now, let the assertion be true for all points of M^0 with at most t binding inequalities, and assume that $\#I(z^0) = t+1$. Then, in a neighborhood of z^0, one has $\#I(z) \leq t+1$. At this point, Lemma 2.2 comes into play as the finite version of Lemma 2.1. Accordingly, one may suppose that, in the new local C^1-coordinates $y=\psi(z)$, the point z^0 is mapped onto the origin $0\in\mathbb{R}^n$ and that the constraint set M is locally described by the system of (in)equalities.

$$y_1\geq\phi_i(y_2,\ldots,y_{n-p}),\quad i=1,\ldots,t+1, \tag{9a}$$

$$y_{n-p+1}=\cdots=y_n=0, \tag{9b}$$

where $\phi_i\in C^1(\mathcal{O}(0),\mathbb{R})$ and $\mathcal{O}(0)$ is an open neighborhood of $0\in\mathbb{R}^{n-p-1}$. Next, define

$$y^{\#}:=(y_2,\ldots,y_{n-p}),\qquad I^*(y^{\#}):=\{i\mid\phi_i(y^{\#})=\max_{j=1,\ldots,t+1}\phi_j(y^{\#})\}.$$

There are two cases to be distinguished.

Case 1. There is a neighborhood $\mathcal{O}_1(0)\subseteq\mathbb{R}^{n-p-1}$ such that

$$I^*(y^{\#})=I^*(0)=\{1,\ldots,t+1\},\qquad\forall y^{\#}\in\mathcal{O}_1(0).$$

In this case, one single inequality [e.g., $y_1-\phi_1(y^{\#})\geq 0$], together with the equations $y_{n-p+1}=\cdots=y_n=0$, yield an equivalent description of (9), the other inequalities in (9) being redundant. Evidently, this reduced description fulfills LICQ and it leads, in the old coordinates, to a reduced local description of M near z^0 which also satisfies LICQ, since linear independence is invariant under local diffeomorphisms. Consequently, $z^0\in A$.

Case 2. The first case is not met. Then, arbitrarily near $0\in\mathbb{R}^{n-p-1}$, there is a point $(y^a)^{\#}$ with $I^*((y^a)^{\#})\leq t$. Now, consider the point

$$y^a:=(y_1^a,(y^a)^{\#},0),\qquad\text{where } y_1^a:=\max_{i=1,\ldots,t}\phi_i((y^a)^{\#})\text{ and } 0\in\mathbb{R}^p.$$

By continuity of the function $\max_{i=1,\ldots,t}\phi_i$, one may choose $y^a\in\mathbb{R}^n$ arbitrarily near $0\in\mathbb{R}^n$. But now, the induction hypothesis applies to y^a, the number of binding inequalities being less than or equal to t. Accordingly, one finds arbitrarily near y^a, a feasible point y^b such that, in a neighborhood of y^b, a local redescription of the constraint set may be found which satisfies LICQ. Since y^b is arbitrarily near $0\in\mathbb{R}^n$, this means that, in the old coordinates, arbitrarily near z^0 there exists a point $\psi^{-1}(y^b)\in A$. Note that density carries over by the diffeomorphism ψ from Lemma 2.2. This terminates the induction step. □

3. Concluding Remarks

The proof of Theorem 2.1 confirms that, under the Mangasarian–Fromovitz constraint qualification, the set of feasible points with binding inequalities contains an open and dense subset where one single inequality together with all equality constraints are sufficient locally to describe the feasible set. Furthermore, this local description fulfills the linear independence constraint qualification. This fact might be of interest for structural investigations, for instance in parametric optimization. Roughly speaking, one can argue that weakening of linear independence to the Mangasarian–Fromovitz constraint qualification has severe consequences for only very few points of the feasible set. There is yet another interpretation of Theorem 2.1. Under the Mangasarian–Fromovitz constraint qualification, the set of feasible points with binding inequalities in finite optimization may be described, in suitable C^1-coordinates, by means of a locally Lipschitz-continuous function $\phi: \mathcal{O} \to \mathbb{R}$ ($\mathcal{O}$ an open subset of $\mathbb{R}^{n-1}$), which is differentiable not only on a set of full Lebesgue measure (Rademachers theorem), but even on a set which is open and dense in $\mathcal{O}$. This excludes the set of nondifferentiability points of ϕ to be dense in $\mathcal{O}$.

A straightforward generalization of these results to semi-infinite optimization (using Lemma 2.1, instead of Lemma 2.2) is not possible, because the technique of induction does not apply.

References

1. GUDDAT, J., and JONGEN, H. T., *On Global Optimization Based on Parametric Optimization*, Advances in Mathematical Optimization, Edited by J. Guddat *et al.*, Akademie-Verlag, Berlin, Germany, Vol. 45, pp. 63–79, 1988.
2. GAUVIN, J., and TOLLE, J. W., *Differential Stability in Nonlinear Programming*, SIAM Journal on Control and Optimization, Vol. 15, pp. 294–311, 1977.
3. KOJIMA, M., *Strongly Stable Stationary Solutions in Nonlinear Programs*, Analysis and Computation of Fixed Points, Edited by S. M. Robinson, Academic Press, New York, New York, pp. 93–138, 1980.
4. GUDDAT, J., JONGEN, H. T., and RUECKMANN, J., *On Stability and Stationary Points in Nonlinear Optimization*, Journal of the Australian Mathematical Society, Series B, Vol. 28, pp. 36–56, 1986.
5. SCHECTER, S., *Structure of the First-Order Solution Set for a Class of Nonlinear Programs with Parameters*, Mathematical Programming, Vol. 34, pp. 84–110, 1986.
6. JONGEN, H. T., JONKER, P., and TWILT, F., *Parametric Optimization: The Kuhn–Tucker Set*, Parametric Optimization and Related Topics, Edited by J. Guddat *et al.*, Akademie-Verlag, Berlin, Germany, Vol. 35, pp. 196–208, 1987.
7. JONGEN, H. T., TWILT, F., and WEBER, G. W., *Semi-Infinite Optimization: Structure and Stability of the Feasible Set*, Memorandum No. 838, University of Twente, Twente, Holland, 1989.
8. CLARKE, F. H., *Optimization and Nonsmooth Analysis*, Wiley, New York, New York, 1983.