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ORIGINAL PAPER

An agricultural investment problem subject to probabilistic constraints

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Abstract

Motivated by a model introduced by Moiseev, we consider a problem of optimal investment into agricultural infrastructure (irrigation, storage) under uncertainty (demand, yield coefficients of soil). Unlike the risk-neutral approach of Moiseev, we formulate a risk-averse model based on joint probabilistic or chance constraints. We assume the random vector to obey a continuous Gaussian distribution. The probabilities of satisfying the demand of cereals and of not wasting excess harvest up to some given thresholds are calculated in dependence on the investment decisions in a multi-period setting. Numerical results are presented for a small-dimensional example.

Keywords Reservoir management · Probabilistic constraints · Agricultural investment · Optimization with chance constraints

Mathematics Subject Classification 90B05 · 90C15

1 Introduction

Reservoir problems play an important role in diverse applications. One may think of water reservoirs in flood protection or generation of hydro energy, of battery storages in cars or minigrids, of bank accounts or pension funds in finance or of silos as agricultural storages. Reservoirs serve to balance discrepancies between the offer and demand of a certain good. Typically, in the management of reservoirs, some objective function

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of the taken decision (e.g., the time profile of releasing the contents of the reservoir) has to be minimized (cost) or maximized (profit) while the decision is subject to certain inequality constraints. The latter may describe, for instance, limits for the filling level in the reservoir or satisfaction of an exterior demand by releasing the reservoir. Most of the optimization problems arising in this way are faced with uncertainties in inflow or release. Examples are precipitation for water reservoirs, random demand of food in supermarkets or solar energy for battery storages. Accordingly, methods of stochastic optimization taking into account the random character of the objective and constraints in the optimization problem, are central in the solution of such problems. Such methods comprise, first of all, multistage stochastic programming (Alais et al. 2017; Pereira and Pinto 1991) and probabilistic programming (Van Ackooij et al. 2014; Loiaciga 1988; Loucks et al. 1981). The methods differ essentially in how they cope with randomness, e.g., risk neutral (expected-value) versus risk averse (e.g., conditional value at risk, probabilistic constraints) models or discrete versus continuous distributions. In this paper, we shall consider a model with probabilistic constraints on the offer-demand relation in the context of an agricultural problem. More precisely, this means that the probabilities of uncovered demand due to insufficient harvest or waste of cereals due to insufficient storage capacities, respectively, are controlled. The particular challenge of this model relies on the *continuou*s character of the random distribution within *joint* probabilistic constraints similar to Van Ackooij et al. (2014). However, an additional special difficulty arises through the presence of hard or *almost sure* constraints (modeling empty or completely filled reservoir). This adds a combinatorial feature in the theoretical derivation and numerical computation of probabilities. Our work is primarily devoted to proper modeling and to proposing a way of computing the involved probabillities. As far as the numerical solution is concerned, we content ourselves to the illustration of a small dimensional example, leaving the application of more efficient approaches for further research. We will also neglect the consideration of dynamic aspects in decision making (optimal decision policies as functions of past random obeservations, as in [4]. This means that our decision process is completely static (‘here-and-now’).

2 An optimal investment problem in agriculture

In the following, we pick up an optimal investment problem in agriculture as presented by Moiseev in (1985, p. 19). We assume that for a given future period of T years certain investment capitals are given. Each year, the capital can be invested either in endowing new agricultural areas with irrigation facilities (thus improving the yield of soil in dry periods) or in constructing silos for the storage of cereal (thus creating reserves for years with low harvest or high demand in cereals). The yields of dry and irrigated lands as well as the demand of cereals will be considered to be (time-dependent) random variables. Investments are to be made as to minimize both a cereal surplus (amount of cereals that cannot be stored due to fully employed silos and thus are wasted) as well as cereal deficit or negative surplus (demand of cereals that cannot be met and thus has to be imported at possibly high costs). Contrary to Moiseev (1985), we won’t treat the random character of the problem in a risk-neutral way by means of expectation-based criteria but in a probabilistic, thus risk averse fashion.

2.1 Parameters of the problem

For each year $t = 1, \dots, T$, we consider the following state variables:

$$\begin{aligned}
\xi_t &= \text{yield coefficient without irrigation}\\
\eta_t &= \text{yield coefficient with irrigation}\\
\delta_t &= \text{annual cereal demand}\\
h_t &= \text{annual cereal harvest}\\
\rho_t &= \text{non-irrigated agricultural area}\\
s_t &= \text{irrigated agricultural area}\\
q_t &= \text{amount of cereals added to silos (negative: withdrawn from silos)}\\
g_t &= \text{total capacity of silos}\\
r_t &= \text{total amount of cereals stored in silos}\\
\Delta_t &= \text{cereal surplus (wasted harvest if positive, shortage if negative),}
\end{aligned}$$

the control variables:

$$\begin{aligned}
x_t &= \text{capital invested in the construction of silos}\\
y_t &= \text{capital invested in irrigation}
\end{aligned}$$

and the data:

$$\begin{aligned}
S &= \text{total agricultural area}\\
c &= \text{unit costs for elevating silos per unit}\\
d &= \text{unit costs for irrigation}\\
z_t &= \text{total annual investment.}
\end{aligned}$$

The initial values of the problem are given by:

$$s_0, g_0, r_0 \quad \left(0 \le s_0 \le s,\ 0 \le r_0 \le g_0\right).$$

All quantities are related by the equations ($t = 1, \dots, T$):

$$\begin{aligned}
S &= \rho_t + s_t\\
h_t &= \xi_t \rho_t + \eta_t s_t\\
g_t &= g_{t-1} + c^{-1} x_t\\
s_t &= s_{t-1} + d^{-1} y_t\\
z_t &= x_t + y_t\\
q_t &= \begin{cases} \min\left\{h_t - \delta_t, g_t - r_{t-1}\right\} & \text{if } h_t \ge \delta_t\\ \max\left\{h_t - \delta_t, -r_{t-1}\right\} & \text{if } h_t < \delta_t \end{cases}\\
r_t &= r_{t-1} + q_t\\
\Delta_t &= h_t - \delta_t - q_t.
\end{aligned}$$

Note that in the relation for q_t it is taken into account that in the case of the harvest exceding the demand, the difference $h_t - \delta_t$ can be stored in the silos but only up to the free capacity $g_t - r_{t-1}$, while in the opposite case cereals can be taken from the silos to cover the deficit but only up to the currently stored amount r_{t-1}.

Since the total annual investment z_t is given, it follows from $z_t = x_t + y_t$ that the capital invested in the construction of silos $x := (x_1, \dots, x_T)$ remains the only control variable.

The total area of land S being given, one has to add the constraint $s_t \le S$ for all $t = 1, \dots, T$ on the state variable s_t. This entails the constraint

$$0 \le y_t \le d(S - s_{t-1}) \quad (t = 1, \dots, T).$$

on y_t and finally the constraint

$$z_t \ge x_t \ge \max\{z_t - d(S - s_{t-1}), 0\} \quad (t = 1, \dots, T) \tag{1}$$

on the remaining control variable.

We shall consider the variables ξ, η, δ (yield coefficients and demand) as random parameters defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and obeying a joint Gaussian distribution with mean μ and regular covariance matrix Σ:

$$\sigma := (\xi, \eta, \delta) := (\xi_t, \eta_t, \delta_t)_{t=1}^T \sim \mathcal{N}(\mu, \Sigma). \tag{2}$$

2.2 Reduced model after elimination of state variables

For all $t = 1, \dots, T$ one has that

$$\begin{aligned}
s_t &= s_{t-1} + d^{-1}(z_t - x_t) = s_0 + d^{-1} \underbrace{\sum_{\tau=1}^{t} z_\tau}_{Z_t} - d^{-1} \sum_{\tau=1}^{t} x_\tau = s_0 + Z_t - d^{-1} \sum_{\tau=1}^{t} x_\tau \\
h_t &= \xi_t (S - s_t) + \eta_t s_t \\
&= \xi_t \left(S - s_0 + d^{-1} \sum_{\tau=1}^{t} x_\tau - Z_t \right) + \eta_t \left(s_0 + Z_t - d^{-1} \sum_{\tau=1}^{t} x_\tau \right) \\
g_t &= g_{t-1} + c^{-1} x_t = g_0 + c^{-1} \sum_{\tau=1}^{t} x_\tau \\
r_t &= r_{t-1} + \begin{cases} \min\{h_t - \delta_t, g_t - r_{t-1}\} & \text{if } h_t \ge \delta_t \\ \max\{h_t - \delta_t, -r_{t-1}\} & \text{if } h_t < \delta_t \end{cases} \\
&= \begin{cases} \min\{r_{t-1} + h_t - \delta_t, g_t\} & \text{if } h_t \ge \delta_t \\ \max\{r_{t-1} + h_t - \delta_t, 0\} & \text{if } h_t < \delta_t \end{cases} \\
&= \begin{cases} 0 & \text{if } h_t - \delta_t \in (-\infty, -r_{t-1}] \\ r_{t-1} + h_t - \delta_t & \text{if } h_t - \delta_t \in (-r_{t-1}, g_t - r_{t-1}] \\ g_t & \text{if } h_t - \delta_t \in (g_t - r_{t-1}, \infty) \end{cases}
\end{aligned} \tag{3}$$

With the formula for s_t one may rewrite the constraint (1) in the explicit form:

$$0 \leq x_t \leq z_t; \quad \sum_{\tau=1}^{t} x_\tau \geq \sum_{\tau=1}^{t} z_\tau - d(S - s_0) \quad (t = 1, \ldots, T). \tag{4}$$

3 Probabilistic model

Using the reduced model above, one observes that finally the cereal surplus may be written as a function $\Delta_t(x, \sigma)$ depending on the control x and on the random parameter σ only. Obviously, the objective consists in minimizing - in a stochastic sense to be made precise - the norm of the vector

absolute value of the cereal surplus (i.e., unused amount of harvested cereals or non-satisfied demand) over the whole time horizon. This amounts to minimizing the random quantity

$$\Delta^{\max}(x, \sigma) := \max_{t=1,\ldots,T} \left|\Delta_t(x, \sigma)\right|.$$

To give sense to such an objective, one has to free it from the concrete unknown outcomes of the random parameter. The simplest way to do so would consist in replacing the random parameter σ by its expectation $\mathbb{E}\sigma$. One would arrive at the problem

$$\min_{x \in X} \Delta^{\max}(x, \mathbb{E}\sigma),$$

where the set X refers to the constraints (4). On a slightly more advanced level, one could minimize the expected maximum of the absolute value of cereal surplus over the whole period of time which leads to the problem

$$\min_{x \in X} \mathbb{E}\Delta^{\max}(x, \sigma). \tag{5}$$

This last and several other expectation-based criteria to be minimized have been proposed in (Moiseev 1985, p.21-22) in the context of the presented reservoir model. Note, however, that expectation-based criteria are just risk-neutral but not risk averse. This means that they are performing well when randomness realizes itself close to the average. On the other hand, the solutions found may turn out to be poor if this not the case, so there is a substantial risk that the absolute value of the cereal surplus is often much larger under the calculated optimal decision x than predicted for mean values of the random parameter. This is why a probabilistic model seems to be a natural remedy here. The following two probabilistic optimization problems might be of interest then:

- Maximisation of the probability that the absolute value of cereal surplus stays smaller than some given $\varepsilon > 0$ throughout the time horizon:

$$\max_{x} \mathbb{P}(\Delta^{\max}(x, \sigma) \leq \varepsilon). \tag{6}$$

- Minimisation of the total sum of investments under the constraint that the probability of the absolute value of cereal surplus staying smaller than some given $\varepsilon > 0$ throughout the time horizon be larger than a given value $p \in [0, 1]$:

$$\min_{x,z} \left\{ \sum_{t=1}^{T} z_t \mid \mathbb{P}(\Delta^{\max}(z, x, \sigma) \le \varepsilon) \ge p \right\} \tag{7}$$

The difference between both problems consists in assuming the total investment process to be given or to be free and subject to optimization. In the latter case the quantity $\Delta^{\max}(z, x, \sigma)$ also depends on the investment variable z. Even if one is primarily interested in (7), it makes sense to solve (6) first, because its optimal value provides a maximum probability $p^{\max}$ that can be achieved at all. This means that specifying a desired level $p > p^{\max}$ in (7) would result in an empty set of feasible decisions.

A certain shortcoming of the previous models lies on the fact that the signs of cereal surplus are given the same meaning. However, there exists a difference in the interpretation of positive cereal surplus (wasted harvest) and negative "surplus" (unsatisfied demand of cereals). Indeed, the latter may be considered the more critical of the two situations. This suggests to treat both aspects separately and to consider the following problem:

$$\max_{x} \min \left\{ \mathbb{P}\big(\Delta_t(x, \sigma) \le \varepsilon_1 \ \forall t = 1, \dots, T\big), \mathbb{P}\big(\Delta_t(x, \sigma) \ge -\varepsilon_2 \ \forall t = 1, \dots, T\big) \right\}. \tag{8}$$

In this model, one chooses two different tolerances $\varepsilon_1, \varepsilon_2$ for keeping the lost harvest and the harvest deficit small throughout the time horizon. Maximizing both probabilities for doing so simultaneously is equivalent to maximizing the minimum of them. Accordingly, it makes sense to modify the free investment problem (7) as

$$\min_{x,z} \left\{ \sum_{t=1}^{T} z_t \mid \min \left\{ \mathbb{P}\big(\Delta_t(x, \sigma) \le \varepsilon_1 \ \forall t = 1, \dots, T\big), \mathbb{P}\big(\Delta_t(x, \sigma) \ge -\varepsilon_2 \ \forall t = 1, \dots, T\big) \right\} \ge p \right\}. \tag{9}$$

3.1 Single-year-problem

We shall start our considerations of problem (8) with a single timestep $T = 1$. Although this is certainly of limited practical interest, this problem allows—contrary to several timesteps—for explicit formulae of the underlying probabilities and thus for elementary numerical solutions. At the same time the case distinction with respect to silos being empty, partially or totally filled already points to the direction of how to calculate probabilities for several time steps. The main task consists in making the probabilities

$$\begin{aligned} \mathbb{P}\big(\Delta_1(x_1, \sigma_1) \le \varepsilon_1\big) &= \mathbb{P}\big(\Delta_1(x_1, \xi_1, \eta_1, \delta_1) \le \varepsilon_1\big) \\ \mathbb{P}\big(\Delta_1(x_1, \sigma_1) \ge -\varepsilon_2\big) &= \mathbb{P}\big(\Delta_1(x_1, \xi_1, \eta_1, \delta_1) \ge -\varepsilon_2\big). \end{aligned}$$

more explicit. To this aim, we introduce according to the representation of r_t obtained above the four random events:

$$\begin{aligned} A_1 &:= \{\omega \in \Omega | h_1(\omega) - \delta_1(\omega) \in (-\infty, -r_0]\} \\ B_1 &:= \{\omega \in \Omega | h_1(\omega) - \delta_1(\omega) \in [0, g_1 - r_0]\} \\ C_1 &:= \{\omega \in \Omega | h_1(\omega) - \delta_1(\omega) \in (-r_0, 0)\} \\ D_1 &:= \{\omega \in \Omega | h_1(\omega) - \delta_1(\omega) \in (g_1 - r_0, \infty)\} \end{aligned} \tag{10}$$

With the definition q_t and the representation of g_t derived earlier, one arrives at

$$\Delta_1(x_1, \sigma_1(\omega)) = \begin{array}{cl} h_1(\omega) - \delta_1(\omega) + r_0 & \text{if } \omega \in A_1 \\ 0 & \text{if } \omega \in B_1 \\ 0 & \text{if } \omega \in C_1 \\ h_1(\omega) - \delta_1(\omega) + r_0 - g_0 - c^{-1}x_1 & \text{if } \omega \in D_1 \end{array}.$$

Evidently the case distinction in (10) defines a partition of the whole event set Ω. Therefore,

$$\begin{aligned} \mathbb{P}(\Delta_1(x_1, \sigma_1) \le \varepsilon_1) = {} & \mathbb{P}(\omega \in A_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1) + \mathbb{P}(\omega \in B_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1) \\ & + \mathbb{P}(\omega \in C_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1) + \mathbb{P}(\omega \in D_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1). \end{aligned}$$

One gets that

$$\begin{aligned} \mathbb{P}(\omega \in A_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1) &= \mathbb{P}(h_1 - \delta_1 \le -r_0) \\ \mathbb{P}(\omega \in B_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1) &= \mathbb{P}(0 \le h_1 - \delta_1 \le g_0 + c^{-1}x_1 - r_0) \\ \mathbb{P}(\omega \in C_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1) &= \mathbb{P}(-r_0 < h_1 - \delta_1 < 0) \\ \mathbb{P}(\omega \in D_1, \Delta_1(x_1, \sigma_1(\omega)) \le \varepsilon_1) &= \mathbb{P}(g_0 + c^{-1}x_1 - r_0 < h_1 - \delta_1 \le \varepsilon_1 + g_0 + c^{-1}x_1 - r_0) \end{aligned}$$

Finally, it follows from the previous expressions that

$$\mathbb{P}(\Delta_1(x_1, \sigma_1) \le \varepsilon_1) = \mathbb{P}(h_1 - \delta_1 \le \varepsilon_1 + g_0 + c^{-1}x_1 - r_0)$$

which by the expansion we made for h_t, amounts to

$$\begin{aligned} & \mathbb{P}(\Delta_1(x_1, \sigma_1) \le \varepsilon_1) = \\ & \mathbb{P}(\xi_1 \cdot (S - s_0 - d^{-1}z_1 + d^{-1}x_1) + \eta_1 \cdot (s_0 + d^{-1}z_1 - d^{-1}x_1) - \delta_1 \le \varepsilon_1 + g_0 + c^{-1}x_1 - r_0) = \\ & \mathbb{P}(\sigma_1^T B(x_1) \le \varepsilon_1 + g_0 + c^{-1}x_1 - r_0) \end{aligned}$$

where

$$\sigma_1 := (\xi_1, \eta_1, \delta_1); \quad B(x_1) := \begin{pmatrix} S - s_0 - d^{-1}z_1 + d^{-1}x_1 \\ s_0 + d^{-1}z_1 - d^{-1}x_1 \\ -1 \end{pmatrix}.$$

Given the parameters of σ_1 (see (2)), one gets that

$$\sigma_1^T B(x_1) \sim \mathcal{N}(B(x_1)^T \mu, B(x_1)^T \Sigma B(x_1)).$$

Hence,

$$\frac{B(x_1)^T(\sigma_1-\mu)}{\sqrt{B(x_1)^T\Sigma B(x_1)}} \sim \mathcal{N}(0,1)$$

and consequently,

$$\mathbb{P}\big(\Delta_1\big(x_1,\sigma_1\big) \le \varepsilon_1\big) =$$

$$\mathbb{P}\left(\frac{B(x_1)^T(\sigma_1-\mu)}{\sqrt{B(x_1)^T\Sigma B(x_1)}} \le \frac{\varepsilon_1+g_0+c^{-1}x_1-r_0-B(x_1)^T\mu}{\sqrt{B(x_1)^T\Sigma B(x_1)}}\right) =$$

$$\Phi\left(\frac{\varepsilon_1+g_0+c^{-1}x_1-r_0-B(x_1)^T\mu}{\sqrt{B(x_1)^T\Sigma B(x_1)}}\right).$$

where Φ denotes the distribution function of the standard Gaussian law $\mathcal{N}(0,1)$. Similarly, one calculates

$$\begin{aligned}\mathbb{P}\big(\Delta_1\big(x_1,\sigma_1\big) \ge -\varepsilon_2\big) &= \mathbb{P}\big(h_1-\delta_1 \ge -r_0-\varepsilon_2\big)\\ &= \mathbb{P}\big(\xi_1\cdot\big(S-s_0-d^{-1}z_1+d^{-1}x_1\big)+\eta_1\cdot\big(s_0+d^{-1}z_1-d^{-1}x_1\big)-\delta_1 \ge -r_0-\varepsilon_2\big)\\ &= \mathbb{P}\big(\sigma_1^T B(x_1) \ge -r_0-\varepsilon_2\big).\end{aligned}$$

From here it results that

$$\begin{aligned}\mathbb{P}\big(\Delta_1\big(x_1,\sigma_1\big) \ge -\varepsilon_2\big) &= \mathbb{P}\left(\frac{B(x_1)^T(\sigma_1-\mu)}{\sqrt{B(x_1)^T\Sigma B(x_1)}} \ge \frac{-r_0-\varepsilon_2-B(x_1)^T\mu}{\sqrt{B(x_1)^T\Sigma B(x_1)}}\right)\\ &= 1-\Phi\left(\frac{-r_0-\varepsilon_2-B(x_1)^T\mu}{\sqrt{B(x_1)^T\Sigma B(x_1)}}\right).\end{aligned}$$

Summarizing, one ends up at

$$\min\big\{\mathbb{P}\big(\Delta_1(x,\sigma) \le \varepsilon_1\big), \mathbb{P}\big(\Delta_1(x,\sigma) \ge -\varepsilon_2\big)\big\} =$$

$$\min\left\{\Phi\left(\frac{\varepsilon_1+g_0+c^{-1}x_1-r_0-B(x_1)^T\mu}{\sqrt{B(x_1)^T\Sigma B(x_1)}}\right), 1-\Phi\left(\frac{-r_0-\varepsilon_2-B(x_1)^T\mu}{\sqrt{B(x_1)^T\Sigma B(x_1)}}\right)\right\}.$$

This last function is easily implemented in standard codes so that problem (8) can be solved in a straightforward manner by one-dimensional optimization for $T=1$.

As a numerical illustration, we consider a problem with the following values:

$$S = 60, s_0 = r_0 = g_0 = 0, z_1 = 35000, c = 15, d = 1000, \varepsilon_1 = 200, \varepsilon_2 = 100,$$

$$\mu = (30, 60, 2500), \quad \Sigma = \begin{pmatrix} 25 & 8 & 0 \\ 8 & 4 & 0 \\ 0 & 0 & 100000 \end{pmatrix}$$

The base units used here are Euro (costs), centner = 50 kilograms for mass of cereals and hectare = 10.000 square meters for agricultural areas. All derived quantities (yield coefficients, unit costs) are to be interpreted with respect to these units. The two probability functions occuring inside (8) and referring to keeping the cereal surplus smaller than ε_1 and the deficit smaller than ε_2 are illustrated in Fig. 1.

The minimum of these two functions achieves its maximum at $x \approx 4737$ and realizes a probability of $p \approx 0.795$.

3.2 Problem with several years

Following the previous derivations, it makes sense to introduce for each $t = 1, \dots, T$ the following event sets:

$$\begin{aligned} A_{t,1} &:= \left\{ \omega \in \Omega | h_t(\omega) - \delta_t(\omega) \in \left(-\infty, -r_{t-1}(\omega) \right] \right\} \\ A_{t,2} &:= \left\{ \omega \in \Omega | h_t(\omega) - \delta_t(\omega) \in \left[0, g_t - r_{t-1}(\omega) \right] \right\} \\ A_{t,3} &:= \left\{ \omega \in \Omega | h_t(\omega) - \delta_t(\omega) \in \left(-r_{t-1}(\omega), 0 \right) \right\} \\ A_{t,4} &:= \left\{ \omega \in \Omega | h_t(\omega) - \delta_t(\omega) \in \left(g_t - r_{t-1}(\omega), \infty \right) \right\}. \end{aligned} \tag{11}$$

Here, the notation $r_{t-1}(\omega)$ indicates that the quantity of cereals in the silos is random by following the dynamics (3) starting with the deterministic initial value r_0. One obtains for each $t = 1, \dots, T$ that

$$\Delta_t(x, \sigma(\omega)) = \begin{cases} h_t(\omega) - \delta_t(\omega) + r_{t-1}(\omega) & \text{if } \omega \in A_{t,1} \\ 0 & \text{if } \omega \in A_{t,2} \\ 0 & \text{if } \omega \in A_{t,3} \\ h_t(\omega) - \delta_t(\omega) + r_{t-1}(\omega) - g_t & \text{if } \omega \in A_{t,4} \end{cases}.$$

The following relation is evident:

$$\left\{ \omega \in \Omega | \Delta_t(x, \sigma(\omega)) \le \varepsilon_1 \right\} = \bigcup_{i=1}^{4} \left\{ \omega \in A_{t,i} | \Delta_t(x, \sigma(\omega)) \le \varepsilon_1 \right\} \quad \forall t = 1, \dots, T.$$

One observes that

$$\Delta_t(x, \sigma(\omega)) \le 0 \le \varepsilon_1 \quad \forall \omega \in A_{t,1} \cup A_{t,2} \cup A_{t,3} \ \forall t = 1, \dots, T.$$

Moreover,

$$B_t := A_{t,4} \cap \left\{ \omega \in \Omega | \Delta_t(x, \sigma(\omega)) \le \varepsilon_1 \right\} = \left\{ \omega \in \Omega | 0 < h_t(\omega) - \delta_t(\omega) + r_{t-1}(\omega) - g_t \le \varepsilon_1 \right\}$$

for all $t = 1, \dots, T$. It follows that

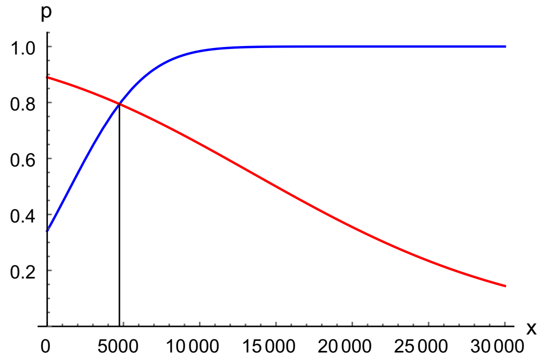

Fig. 1 Plot of probability to keep cereal surplus smaller than ε_1 (blue) and cereal deficit smaller than ε_2 (red) for a single year (data see text) over capital x invested for the construction of silos (color figure online)

$$\begin{aligned}\left\{\omega \in \Omega | \Delta_t(x, \sigma(\omega)) \leq \varepsilon_1\right\} &= A_{t,1} \cup A_{t,2} \cup A_{t,3} \cup B_t \\ &= \left\{\omega \in \Omega | h_t(\omega) - \delta_t(\omega) + r_{t-1}(\omega) - g_t \leq \varepsilon_1\right\}\end{aligned} \tag{12}$$

for all $t = 1, \ldots, T$. In a similar manner, one shows that

$$\left\{\omega \in \Omega | \Delta_t(x, \sigma(\omega)) \geq -\varepsilon_2\right\} = \left\{\omega \in \Omega | h_t(\omega) - \delta_t(\omega) + r_{t-1}(\omega) \geq -\varepsilon_2\right\} \quad \forall t = 1, \ldots, T. \tag{13}$$

In order to finally calculate the desired probabilities, it remains to make precise the quantities $r_t(\omega)$. Let us consider an event

$$\omega \in A_{1,i_1} \cap A_{2,i_2} \cdots \cap A_{T,i_T} \tag{14}$$

for some arbitrary choice

$$\left\{i_1, \ldots, i_T\right\} \in \{1, \ldots, 4\}^T.$$

For this scenario, the dynamics (3) assigns a sequence $\left\{r_0, r_1(\omega), \ldots, r_T(\omega)\right\}$ according to the rule

$$r_t(\omega) = \begin{cases} r_{t-1}(\omega) + h_t(\omega) - \delta_t(\omega) & \text{if } i_t \in \{2, 3\} \\ 0 & \text{if } i_t = 1 \\ g_t & \text{if } i_t = 4. \end{cases}$$

To make this expression more explicit, we observe that the value of $r_t(\omega)$ depends only on the dynamics starting with the last preceding deterministic value in the sequence, i.e., r_0, 0 or g_t. That is why it makes sene to introduce the following variables

$$\alpha_t^{\{i_1,\ldots,i_t\}} := \begin{cases} \max\left\{\tau \leq t | i_\tau \in \{1, 4\}\right\} & \text{if } \exists \tau \leq t : i_\tau \in \{1, 4\} \\ 0 & \text{otherwise} \end{cases}$$

for all $\left\{i_1, \ldots, i_t\right\} \in \{1, \ldots, 4\}^t$ and all $t = 1, \ldots, T$. Moreover, we define the deterministic initial values of r_t at time $\alpha_t^{\{i_1,\ldots,i_t\}}$:

$$\beta_t^{\{i_1,\dots,i_t\}} := \begin{cases} r_0 & \text{if } \alpha_t^{\{i_1,\dots,i_t\}} = 0 \\ 0 & \text{if } \alpha_t^{\{i_1,\dots,i_t\}} > 0 \text{ et } i_{\alpha_t^{\{i_1,\dots,i_t\}}} = 1 \\ g_{\alpha_t^{\{i_1,\dots,i_t\}}} & \text{if } \alpha_t^{\{i_1,\dots,i_t\}} > 0 \text{ et } i_{\alpha_t^{\{i_1,\dots,i_t\}}} = 4 \end{cases}$$

for all $\{i_1, \dots, i_t\} \in \{1, \dots, 4\}^t$ and all $t = 1, \dots, T$. With these definitions, one may write

$$r_t(\omega) = \beta_t^{\{i_1,\dots,i_t\}} + \sum_{\tau=\alpha_t^{\{i_1,\dots,i_t\}}+1}^{t} \left(h_\tau(\omega) - \delta_\tau(\omega)\right) \quad \forall t = 0, \dots, T. \tag{15}$$

where we put $\beta_0 := r_0$ and $\alpha_0 := 0$ in order to recover that $r_0(\omega) = r_0$.

In the following, we present a formula for the desired probability. As the collection of events (14) forms a partition (disjoint union) of the total event set Ω, one may write

$$\begin{aligned} &\mathbb{P}\left(\Delta_t(x, \sigma(\omega)) \le \varepsilon_1 \quad \forall t = 1, \dots, T\right) \\ &= \sum_{\{i_1,\dots,i_T\}\in\{1,\dots,4\}^T}^{T} \mathbb{P}\left(\left\{\omega | \Delta_t(x, \sigma(\omega)) \le \varepsilon_1 \quad \forall t = 1, \dots, T,\ \omega \in A_{1,i_1} \cap A_{2,i_2} \cdots \cap A_{T,i_T}\right\}\right) \\ &= \sum_{\{i_1,\dots,i_T\}\in\{1,\dots,4\}^T}^{T} p_{i_1,\dots,i_T} \end{aligned} \tag{16}$$

where (using (12), (15) and (11))

$$p_{i_1,\dots,i_T} = \mathbb{P}\left(\begin{array}{c} \forall t = 1, \dots, T : \\ h_t(\omega) - \delta_t(\omega) + \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} + \sum\limits_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \left(h_\tau(\omega) - \delta_\tau(\omega)\right) - g_t \le \varepsilon_1 \\ \\ \forall t = 1, \dots, T : \\ h_t(\omega) - \delta_t(\omega) + \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} + \sum\limits_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \left(h_\tau(\omega) - \delta_\tau(\omega)\right) \le 0 \quad \text{if } i_t = 1 \\ 0 \le h_t(\omega) - \delta_t(\omega) \le g_t - \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} - \sum\limits_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \left(h_\tau(\omega) - \delta_\tau(\omega)\right) \quad \text{if } i_t = 2 \\ -\beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} - \sum\limits_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \left(h_\tau(\omega) - \delta_\tau(\omega)\right) < h_t(\omega) - \delta_t(\omega) < 0 \quad \text{if } i_t = 3 \\ h_t(\omega) - \delta_t(\omega) + \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} + \sum\limits_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \left(h_\tau(\omega) - \delta_\tau(\omega)\right) > g_t \quad \text{if } i_t = 4 \end{array}\right).$$

One easily checks that the very first expression is automatically satisfied in cases $i_t = 1, 2, 3$ (for $i_t = 3$, this follows from the evident relation $r_t(\omega) \le g_t$). The case

$i_t = 4$ is the only one to be restricted by this first inequality. Therefore, one may integrate the first expression in the following block as

$$p_{i_1,\dots,i_T} = \mathbb{P}\left(\begin{array}{c} \forall t = 1,\dots,T: \\ h_t(\omega) - \delta_t(\omega) + \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} + \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) \le 0 \quad \text{if } i_t = 1 \\ 0 \le h_t(\omega) - \delta_t(\omega) \le g_t - \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} - \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) \quad \text{if } i_t = 2 \\ -\beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} - \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) < h_t(\omega) - \delta_t(\omega) < 0 \quad \text{if } i_t = 3 \\ g_t < h_t(\omega) - \delta_t(\omega) + \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} + \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) \le g_t + \varepsilon_1 \quad \text{if } i_t = 4 \end{array}\right). \tag{17}$$

One observes that these last expressions are completely explicit with respect to the random and decision variables (after expanding g_t, $h_t(\omega)$ et $\delta_t(\omega)$ in the manner of the last section). From (13) one obtains an expression similar to (16) for the probability

$$\mathbb{P}\big(\Delta_t(x, \sigma(\omega)) \ge -\varepsilon_2 \quad \forall t = 1,\dots,T\big) = \sum_{\{i_1,\dots,i_T\}\in\{1,\dots,4\}^T}^{T} \tilde{p}_{i_1,\dots,i_T} \tag{18}$$

with the elementary probabilities

$$\tilde{p}_{i_1,\dots,i_T} = \mathbb{P}\left(\begin{array}{c} \forall t = 1,\dots,T: \\ -\varepsilon_2 \le h_t(\omega) - \delta_t(\omega) + \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} + \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) \le 0 \quad \text{if } i_t = 1 \\ 0 \le h_t(\omega) - \delta_t(\omega) \le g_t - \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} - \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) \quad \text{if } i_t = 2 \\ -\beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} - \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) < h_t(\omega) - \delta_t(\omega) < 0 \quad \text{if } i_t = 3 \\ h_t(\omega) - \delta_t(\omega) + \beta_{t-1}^{\{i_1,\dots,i_{t-1}\}} + \sum_{\tau=\alpha_{t-1}^{\{i_1,\dots,i_{t-1}\}}+1}^{t-1} \big(h_\tau(\omega) - \delta_\tau(\omega)\big) > g_t \quad \text{if } i_t = 4 \end{array}\right). \tag{19}$$

In order to illustrate this formula, one considers the computation of

$$\mathbb{P}\big(\Delta_1(x, \sigma(\omega)) \le \varepsilon_1,\ \Delta_2(x, \sigma(\omega)) \le \varepsilon_1\big)$$

for two time steps ($T = 2$). One gets the following values for $\alpha_1^{\{i_1\}}$ and $\beta_1^{\{i_1\}}$:

i_1	$\alpha_1^{\{i_1\}}$	$\beta_1^{\{i_1\}}$
1	1	0
2	0	r_0
3	0	r_0
4	1	g_1

For example, one calculates

$$p_{1,1} = \mathbb{P}\begin{pmatrix} h_1(\omega) - \delta_1(\omega) + r_0 \leq 0 \\ h_2(\omega) - \delta_2(\omega) \leq 0 \end{pmatrix},$$

and, similarly,

$$p_{1,2} = \mathbb{P}\begin{pmatrix} h_1(\omega) - \delta_1(\omega) + r_0 \leq 0 \\ 0 \leq h_2(\omega) - \delta_2(\omega) \leq g_2 \end{pmatrix}; \quad p_{1,3} = 0;$$
$$p_{1,4} = \mathbb{P}\begin{pmatrix} h_1(\omega) - \delta_1(\omega) + r_0 \leq 0 \\ g_2 < h_2(\omega) - \delta_2(\omega) \leq g_2 + \varepsilon_1 \end{pmatrix}.$$

Althoughin the specialcaseof $T = 2$ time steps some expressions can be combined, one has to calculate in general all 4^T for each of the two desired total probabilities. In order to organize this calculus, we will rewrite next the elementary probabilities $p_{i_1,\ldots,i_T}$ and $\tilde{p}_{i_1,\ldots,i_T}$ from (17) and (19) in some more compact matrix form:

$$p_{i_1,\ldots,i_T} = \mathbb{P}\big(B_{i_1,\ldots,i_T}(h(\omega) - \delta(\omega)) \leq w_{i_1,\ldots,i_T}\big);$$
$$\tilde{p}_{i_1,\ldots,i_T} = \mathbb{P}\big(\tilde{B}_{i_1,\ldots,i_T}(h(\omega) - \delta(\omega)) \leq \tilde{w}_{i_1,\ldots,i_T}\big).$$

Since the order of matrices $B_{i_1,\ldots,i_T}, \tilde{B}_{i_1,\ldots,i_T}$ and of vectors $w_{i_1,\ldots,i_T}, \tilde{w}_{i_1,\ldots,i_T}$ may vary with the index $\big(i_1, \ldots, i_T\big)$, it is convenient to specify their components for each time step rather than providing an integral formula. One introduces the vectors $a^{(t)}, b^{(t)} \in \mathbb{R}^T$ as

$$a_i^{(t)} := \begin{cases} 1 & \text{if } \alpha_{t-1}^{\{i_1,\ldots,i_{t-1}\}} + 1 \leq i \leq t \\ 0 & \text{otherwise} \end{cases}; \quad b_i^{(t)} := \begin{cases} 1 & \text{if } i = t \\ 0 & \text{otherwise} \end{cases}.$$

At each time step, the matrices $B_{i_1,\ldots,i_T}, \tilde{B}_{i_1,\ldots,i_T}$ and the vectors $w_{i_1,\ldots,i_T}, \tilde{w}_{i_1,\ldots,i_T}$ are augmented - according to the indices i_t - be it by one or two lines and be it by one or two components. More precisely, one arrives at the following table:

i_t	Augmentation of $B_{i_1,\ldots,i_T}$ by	Augmentation of $\tilde{B}_{i_1,\ldots,i_T}$ by	Augmentation of $w_{i_1,\ldots,i_T}$ by	Augmentation of $\tilde{w}_{i_1,\ldots,i_T}$ by
1	$a^{(t)}$	$-a^{(t)}$ and $a^{(t)}$	$-\beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}}$	$\beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}} + \varepsilon_2$ and $-\beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}}$
2	$-b^{(t)}$ and $a^{(t)}$	$-b^{(t)}$ and $a^{(t)}$	0 and $g_t - \beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}}$	0 and $g_t - \beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}}$
3	$-a^{(t)}$ and $b^{(t)}$	$-a^{(t)}$ and $b^{(t)}$	$\beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}}$ and 0	$\beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}}$ and 0
4	$-a^{(t)}$ et $a^{(t)}$	$-a^{(t)}$	$\beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}} - g_t$ and $g_t - \beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}} + \varepsilon_1$	$\beta_{t-1}^{\{i_1,\ldots,i_{t-1}\}} - g_t$

(20)

3.3 Representation of probabilities as a function of decisions

For the solution of our optimisation problem it is essential to explicitly represent the probabilities $p_{i_1,\ldots,i_T}$ and $\tilde{p}_{i_1,\ldots,i_T}$ from the last section as functions of the decision x. One observes that the matrices $B_{i_1,\ldots,i_T}$ and $\tilde{B}_{i_1,\ldots,i_T}$ are constant while the vector h depends on the random and decision variables via

$$h_t(\omega) = \xi_t(\omega)\left(S - s_0 + d^{-1}\sum_{\tau=1}^{t} x_\tau - Z_t \right) + \eta_t(\omega)\left(s_0 + Z_t - d^{-1}\sum_{\tau=1}^{t} x_\tau \right) \quad (t = 1, \ldots, T)$$

and the vectors w and $\tilde{w}$ depend on x because their components depend on g which itself is a function of x. Hence, the probabilities $p_{i_1,\ldots,i_T}$ may be written as

$$p_{i_1,\ldots,i_T}(x) = \mathbb{P}\big(B_{i_1,\ldots,i_T} f(x,\xi,\eta,\delta) \le w(x)\big), \tag{21}$$

where for $t = 1, \ldots, T$,

$$f_t(x,\xi,\eta,\delta) := \xi_t\left(S - s_0 + d^{-1}\sum_{\tau=1}^{t} x_\tau - Z_t \right) + \eta_t\left(s_0 + Z_t - d^{-1}\sum_{\tau=1}^{t} x_\tau \right) - \delta_t \tag{22}$$

and $w(x)$ is defined by w in the table as a function of g (this includes the dependence of β-terms on g) which itself depends on x. Similarly, one gets that

$$\tilde{p}_{i_1,\ldots,i_T}(x) = \mathbb{P}\big(\tilde{B}_{i_1,\ldots,i_T} f(x,\xi,\eta,\delta) \le \tilde{w}(x)\big). \tag{23}$$

3.4 Computation of probabilities using spherical-radial decomposition

Algorithmically, the easiest way to calculate the probabilities $p_{i_1,\ldots,i_T}(x)$ and $\tilde{p}_{i_1,\ldots,i_T}(x)$ for a given decision vector x would be Monte Carlo simulation. To this aim, one generates N samples $\left\{\left(\xi^{(i)},\eta^{(i)},\delta^{(i)}\right)\right\}_{i=1}^{N}$ of the given Gaussian distribution (2) and checks how many of these samples satisfy the corresponding inequality systems (21) and (23), respectively. The percentage with respect to N then provides estimates for the probabilities and thus for the desired total probabilities (16) and (18). However, the variance of Monte Carlo estimates is relatively large and decreases slowly with the sample size. Another approach, promising a strong reduction of variance in the case of elliptically symmetric (in particular: Gaussain) distributions is the so-called *spherical-radial decomposition*. Assume that ξ is a d-dimensional Gaussian random vector with distribution $\xi \sim \mathcal{N}(m,\Sigma)$. Then, the probability of this vector falling into a (measurable set) A can be represented as the spherical integral

$$\mathbb{P}(\xi \in A) = \int_{v\in\mathbb{S}^{d-1}} \mu_\chi(B(v))d\mu_\zeta(v); \quad B(v) := \{r \ge 0 | m + rLv \in A\}, \tag{24}$$

where L is a Cholesky factor of Σ ($\Sigma = LL^T$), μ_χ is the one-dimensional Chi distribution with d degrees of freedom and μ_ζ is the uniform distribution on the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. For not too complicated sets A (for instance, defined by linear or polynomial inequalities) one may efficiently determine the one-dimensional intersection $B(v)$ of A with the ray generated by the vector Lv and emanating from m. The whole integrand $\mu_\chi(B(v))$ is then easily determined using a highly precise numerical approximation of the distribution function F_χ of the one-dimensional Chi distribution with d degrees of freedom. For instance, if

$$B(v) = \bigcup_{i=1}^{i(v)} \left[\alpha_i, \beta_i\right] \quad \left(\alpha_i < \beta_i < \alpha_{i+1}\right) \tag{25}$$

is a decomposition of $B(v)$ into finitely many disjoint intervals, then

$$\mu_\chi(B(v)) = \sum_{i=1}^{i(v)} F_\chi\left(\beta_i - \alpha_i\right).$$

The integral itself in (24) is approximated empirically by a finite sum

$$\mathbb{P}(\xi \in A) \approx N^{-1} \sum_{i=1}^{N} \mu_\chi\left(B\left(v^{(i)}\right)\right),$$

where $\left\{v^{(i)}\right\}_{i=1}^N$ is a finite (low discrepancy) sample of the uniform distribution on the sphere $\mathbb{S}^{d-1}$. In the following we provide a formula for the sets $B(v)$ associated with the probabilities $p_{i_1,\ldots,i_T}(x)$ and $\tilde{p}_{i_1,\ldots,i_T}(x)$ for a fixed decision vector x. For instance, owing to (21), the set $B(v)$ associated with the probability $p_{i_1,\ldots,i_T}(x)$ can be written as

$$B(v) = \left\{r \geq 0 | B_{i_1,\ldots,i_T} f\left(x, m_a + rLv_a, m_b + rLv_b, m_c + rLv_c\right) \leq w(x)\right\},$$

where we use the partitions $m = \left(m_a, m_b, m_c\right)$ and $v = \left(v_a, v_b, v_c\right)$ of the mean vector m and the direction v induced from the partition (ξ, η, δ) of the entire random vector. Recalling (22), we may represent the components f_t of f as

$$\begin{aligned} f_t\left(x, m_a + rLv_a, m_b + rLv_b, m_c + rLv_c\right) = \\ A_t(x)\left(m_{a,t} + rL_t v_a\right) + B_t(x)\left(m_{b,t} + rL_t v_b\right) - \left(m_{c,t} + rL_t v_c\right), \end{aligned}$$

where, for $t = 1, \ldots, T$, $m_{a,t}$ and $m_{b,t}$ are the components of m_a and m_b, respectively, L_t is the t-th row of matrix L and

$$A_t(x) = \left(S - s_0 + d^{-1} \sum_{\tau=1}^{t} x_\tau - Z_t\right); \quad B_t(x) = \left(s_0 + Z_t - d^{-1} \sum_{\tau=1}^{t} x_\tau\right).$$

Now, the i-th inequality in the system defining $B(v)$ writes as

$$\sum_{t=1}^{T} \left(B_{i_1,\ldots,i_T}\right)_{i,t} \left(A_t(x)\left(m_{a,t} + rL_t v_a\right) + B_t(x)\left(m_{b,t} + rL_t v_b\right) - \left(m_{c,t} + rL_t v_c\right)\right) \leq w_i(x)$$

or, shortly:

$$\gamma_i(x) + r\varkappa_i(x) \leq w_i(x), \tag{26}$$

where

$$\begin{aligned} \gamma_i(x) : \quad &= \sum_{t=1}^{T} \left(B_{i_1,\ldots,i_T}\right)_{i,t} \left(A_t(x)m_{a,t} + B_t(x)m_{b,t} - m_{c,t}\right) \\ \varkappa_i(x) : \quad &= \sum_{t=1}^{T} \left(B_{i_1,\ldots,i_T}\right)_{i,t} \left(A_t(x)L_t v_a + B_t(x)L_t v_b - L_t v_c\right). \end{aligned}$$

Now, it is elementary to see that the set of $r \geq 0$ satisfying (26) is given by the intervals

$$\begin{array}{ll} \left[0, \left(w_i(x) - \gamma_i(x)\right)/\varkappa_i(x)\right] & if\, \varkappa_i(x) > 0 \\ \left[\left(w_i(x) - \gamma_i(x)\right)/\varkappa_i(x), \infty\right) & if\, \varkappa_i(x) < 0 \\ [0, \infty) & if\, \varkappa_i(x) = 0 and \gamma_i(x) \leq w_i(x) \\ \emptyset & if\, \varkappa_i(x) = 0 and \gamma_i(x) > w_i(x) \end{array},$$

where the first indicated interval is defined to be empty whenever $\left(w_i(x) - \gamma_i(x)\right)/\varkappa_i(x) < 0$. In this way we get an interval of values $r \geq 0$ satisfying the i-th inequality in the system defining $B(v)$. Hence, the set of values $r \geq 0$ satisfying all inequalities in the system defining $B(v)$ is obtained as the intersection of these intervals, hence $B(v)$ is itself a simple interval (i.e., $i(v) = 1$ for all v in the decomposition (25)). A similar derivation applies to the probabilities $\tilde{p}_{i_1,\ldots,i_T}(x)$. Figure 2 illustrates the variance reducing effect of spherical-radial decomposition when compared to crude Monte Carlo sampling. A further advantage of spherical-radial decomposition over Monte Carlo is that it allows one to obtain a gradient formula for the probability when derived with respect to the decision vector x Van Ackooij and Henrion (2017) . Moreover, this gradient can be represented as a spherical integral similar to that obtained before for the probability itself. This fact makes it possible to simultaneously sample probabilities and gradients which is numerically useful because the determination of r-values described above for a particular direction v does not need to be redone in the gradient estimation.

4 Numerical solution of five-year problems

In this section we are going to illustrate the solutions of the probability maximization problem (8) under fixed total investments and of the free investment problem (9) under probabilistic constraints for a period of $T = 5$ years. Here, the total probabilties compute as the sum of $4^5 = 1024$ elementary probabilities. Spending more computational effort and applying more efficient nonlinear optimization techniques one might slightly drive up the number of years say towards nine or ten but we shall content ourselves here with the illustration of solutions for this more moderate solution. The inherent complexity of case distinctions will anyhow bound T to rather small values. For the following examples, we shall use the values

$$
\begin{aligned}
&S = 60, s_0 = r_0 = g_0 = 0, c = 15, d = 1000, \varepsilon_1 = 1000, \varepsilon_2 = 500,\\
&z = (35, 30, 25, 20, 15) \cdot 10^3\\
&\mu = (30, 60, 2000, 30, 60, 3700, 30, 60, 2000, 30, 60, 1500, 30, 60, 2000)
\end{aligned}
$$

The covariance matrix Σ of order (15, 15) is assumed to be block-diagonal with 5 (yearly) blocks

$$
\begin{pmatrix} 25 & 8 & 0 \\ 8 & 4 & 0 \\ 0 & 0 & \gamma_i \end{pmatrix}, \qquad \gamma_i = (40, 100, 40, 20, 40) \cdot 10^3.
$$

4.1 Solution of the probability maximization problem (8)

Figure 4 illustrates the solution of the probability maximization problem (8). It can be seen that, under the given values, emphasis is laid on a fast irrigation until after two years the total area is irrigated, whereas investments into the construction of silos is retarded.

The maximum probability found with this solutions amounts to $p^{\max} = 0.976$. For a posterior verification of this probability, 100 harvest and demand scenarios were generated under the given multivariate Gaussian distribution of the random vector σ in (2). The resulting scenarios for the cereal surplus show that one of them violates the bound for cereal surplus ($\varepsilon_1 = 1000$) and three of them violate the bound for cereal deficit ($\varepsilon_2 = 500$) which altogether yields a good coincidence with the calculated theoretical probability for satisfying both bounds simultaneously over the

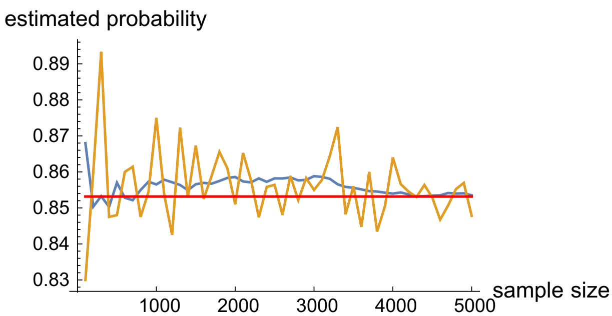

Fig. 2 Comparison of probability estimation via Monte Carlo (orange) and Spherical-Radial Decomposition (blue) as a function of sample size. The red line represents a fairly close approximation to the true probability based on a large sample size (color figure online)

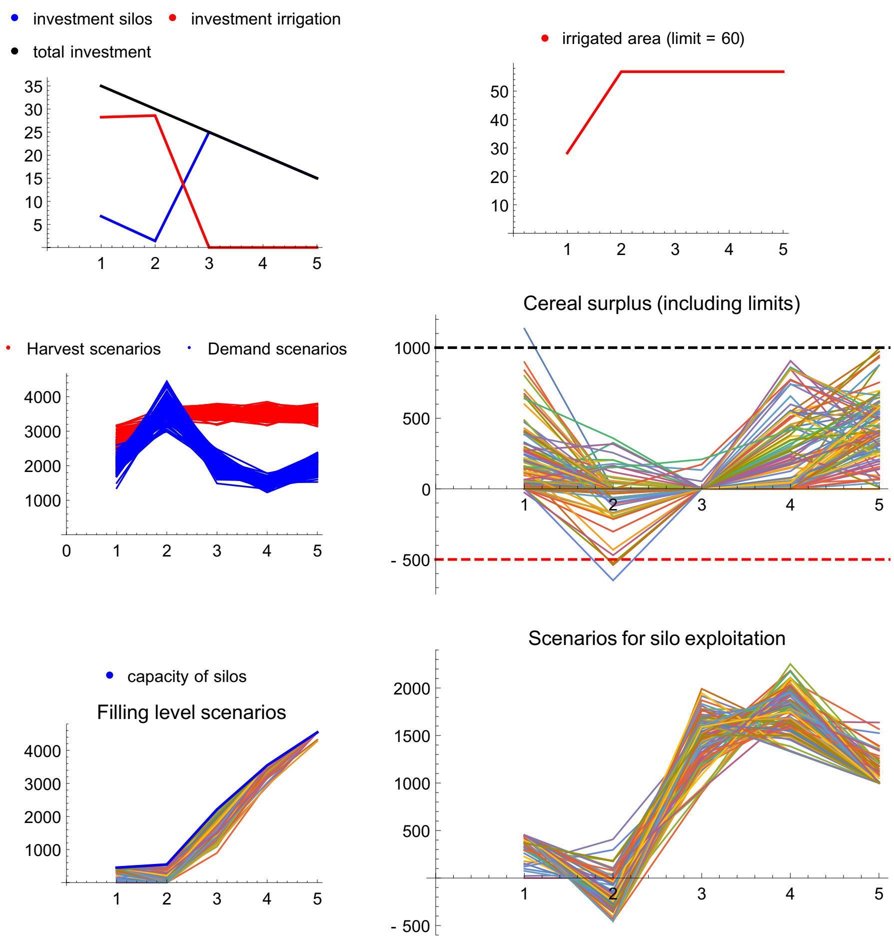

Fig. 3 Illustration of solutions to the probability maximization problem (8) for $T = 5$ years

whole period of five years. It is also seen that all resulting scenarios for the fillilng levels of silos remain below the capacities installed over time. Finally, one observes that silos are emptied only in the second year to keep the deficit low, whereas they are continuously filled otherwise in order to keep the wasted harvest low.

4.2 Solution of the free investment problem (9)

Contrary to the probability maximization under fixed investment policy according to problem (8), we are now going to fix the probability and to find a cost minimal investment policy according to problem (9). So, if we choose a probability level p that is strictly smaller than $p^{\max} = 0.974$ from the solution of problem (8), then we max expect to reach this smaller probability with a smaller total investment.

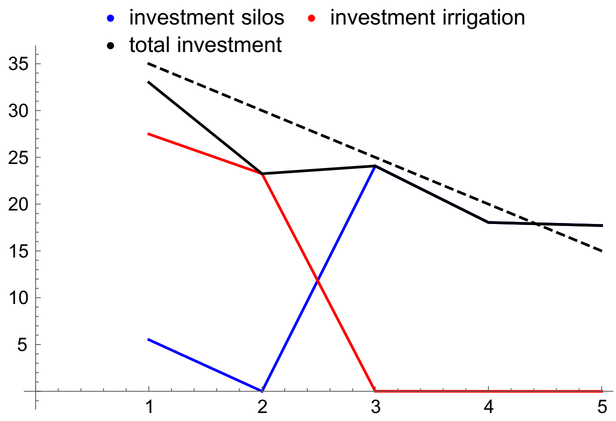

Fig. 4 Illustration of solutions to the free investment problem (9) for $T = 5$ years (dashed: fixed investment policy from problem (8))

Figure 3 illustrates the optimal investment policy and associated irrigation and silo generation policies for a probability $p = 0.9$. The sum of total investments over five years amounts to 116.1, whereas the sum of fixed total investments in problem (8) was 125. In other words, increasing the desired probability from $p = 0.9$ to the maximum possible probability $p^{\max} = 0.974$ comes at the additional costs of 8.9 (approximately 8%).

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