Indexation Strategies and Calmness Constants for Uncertain Linear Inequality Systems

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Dedicated to the memory of Pedro Gil

Abstract The present paper deals with uncertain linear inequality systems viewed as nonempty closed coefficient sets in the $(n + 1)$-dimensional Euclidean space. The perturbation size of these uncertainty sets is measured by the (extended) Hausdorff distance. We focus on calmness constants—and their associated neighborhoods—for the feasible set mapping at a given point of its graph. To this aim, the paper introduces an appropriate indexation function which allows us to provide our aimed calmness constants through their counterparts in the setting of linear inequality systems with a fixed index set, where a wide background exists in the literature.

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1 Introduction

We consider the *uncertain linear inequality system*

$$\left\{ a'x \leq b, \ \begin{pmatrix} a \\ b \end{pmatrix} \in U \right\}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of variables and the *uncertainty set* U is assumed to be a nonempty closed subset of $\mathbb{R}^{n+1}$. All elements in $\mathbb{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$. Accordingly, elements in $\mathbb{R}^{n+1}$ will be written in the form $\binom{u}{v}$, where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$. Observe that (1) is in general a linear *semi-infinite* inequality system (i.e., with finitely many decision variables but possibly infinitely many constraints). Linear semi-infinite inequality systems have been extensively studied in [11].

The uncertainty set U is considered as the parameter to be perturbed. So, formally, we are considering the *parameter space* $CL\left(\mathbb{R}^{n+1}\right)$ of all nonempty closed subsets in $\mathbb{R}^{n+1}$. From the topological side, the space of variables $\mathbb{R}^n$ is endowed with an arbitrary norm $\|\cdot\|$, and the parameter space is equipped with the (extended) Hausdorff distance, d_H, specified in Sect. 2.1 (see e.g. [1] for a comprehensive analysis of the Hausdorff metric).

Associated with the parametrized system (1), roughly speaking referred to as system U, we consider the *feasible set mapping*, $\mathscr{F} : CL\left(\mathbb{R}^{n+1}\right) \rightrightarrows \mathbb{R}^n$, given by

$$\mathscr{F}(U) := \left\{ x \in \mathbb{R}^n \mid a'x \leq b \text{ for all } \begin{pmatrix} a \\ b \end{pmatrix} \in U \right\}.$$

Observe that the closedness assumption on U is not restrictive since the feasible set mapping has the same values if general sets are replaced with their closures, and the same happens with the definition of excess—see (3)—and hence with d_H.

Our main goal consists of providing calmness constants (cf. Sect. 2.2) for $\mathscr{F}$ at a nominal (fixed) element of its graph (U_0, x_0). We can find in the literature different contributions to the calmness of the feasible set mapping in the context of linear systems with a fixed index set T, say $\mathscr{F}^T : \left(\mathbb{R}^{n+1}\right)^T \rightrightarrows \mathbb{R}^n$, which is given by

$$\mathscr{F}^T(\sigma) := \left\{ x \in \mathbb{R}^n \mid a_t'x \leq b_t, \ t \in T \right\}, \tag{2}$$

where

$$\sigma(t) = \begin{pmatrix} a_t \\ b_t \end{pmatrix} \in \mathbb{R}^{n+1}, t \in T.$$

In this framework, the parameter space $\left(\mathbb{R}^{n+1}\right)^T$ is assumed to be endowed with the uniform converge topology; see Sect. 2.1 for details.

With the aim of taking advantage of the vaste literature about calmness for mappings in the format $\mathscr{F}^T$ to derive calmness constants for $\mathscr{F}$, we introduce in Sect. 3.1

a specific *indexation function*, $\mathscr{I} : CL\left(\mathbb{R}^{n+1}\right) \to \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$, which assigns to each set $U \in CL\left(\mathbb{R}^{n+1}\right)$ a certain function $\mathscr{I}_U \in \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$ with rge $\mathscr{I}_U = U$, where rge stands for range (image). In this way, if U is the set of coefficient vectors of system (1), $\sigma = \mathscr{I}_U$ (whose index set is the whole $\mathbb{R}^{n+1}$) can be interpreted as ensentially the same system but with the addition of repeated constraints.

The definition of our indexation function $\mathscr{I}$ is inspired, but sensibly different (see Sect. 3 for details), by the one introduced in [6] with the aim of analyzing the stability of the *optimal value function* of linear optimization problems with uncertain constraints. In the present paper the properties of $\mathscr{I}$ will enable us to derive calmness constants (and associated neighborhoods) for $\mathscr{F}$ from those for $\mathscr{F}^{\mathbb{R}^{n+1}}$. In a second step we wonder whether $\mathbb{R}^{n+1}$ may be replaced with a smaller index set $T \subset \mathbb{R}^{n+1}$.

Paper [4] provides the calmness modulus of $\mathscr{F}^T$ in the particular case when T is finite (see [2] for an extension to the nonlinear case), whereas [5] proves that this calmness modulus is in fact a calmness constant for a certain neighborhood (specified therein) when we restrict oursleves to right-hand-side perturbations. In Sect. 5 of the present paper we show how to extend this result to perturbations of all coefficients. Coming back to our framework of uncertain linear systems, the reader is addressed to [7, 8] for the study of robust local and global error bounds, respectively. Recall that, the local error bound property is closely related to calmness of feasible solutions when only right-hand-side perturbations are allowed. See also [10] for the development of dualy theory in robust linear optimization with infinitely many uncertain constraints.

Now we summarize the main original contributions of the paper. Section 3 motivates (see Example 3.1) and introduces the announced indexation function $\mathscr{I}$ which allows us to derive calmness constants for $\mathscr{F}$ at $(U_0, x_0) \in \text{gph}\mathscr{F}$ (the graph of $\mathscr{F}$) via calmness constants for $\mathscr{F}^{\mathbb{R}^{n+1}}$ at (σ_0, x_0), with $\sigma_0 := \mathscr{I}_{U_0}$. After that, Sect. 4 solves the question of whether or not $\mathbb{R}^{n+1}$ may be replaced with a smaller subset T. Specifically, for $U_0 \subset T \subset \mathbb{R}^{n+1}$, we prove that the calmness of $\mathscr{F}^T$ is equivalent to the calmness of $\mathscr{F}^{\mathbb{R}^{n+1}}$, with the same calmness constants and closely related neighborhoods. We also analyze the particular case when U_0 is the convex hull of some subset in $\mathbb{R}^{n+1}$. Finally, Sect. 5 allows to derive from [5] operative point-based expressions (in terms of the nominal data) for a tight calmness constant for $\mathscr{F}$ and a neighborhood where it works.

2 Preliminaries

Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by conv$X$ and cone$X$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that cone$X$ always contains the zero-vector 0_k, in particular cone$(\emptyset) = \{0_k\}$. If X is a subset of any topological space, int$X$, cl$X$ and bd$X$ stand, respectively, for the *interior*, the *closure* and the *boundary* of X.

2.1 Hausdorff and Chebyshev Distances

The space $CL\left(\mathbb{R}^{n+1}\right)$ will be endowed with the (extended) Hausdorff distance d_H : $CL\left(\mathbb{R}^{n+1}\right) \times CL\left(\mathbb{R}^{n+1}\right) \to [0, +\infty]$ given by

$$d_H\left(U_1, U_2\right) := \max\{e\left(U_1, U_2\right), e\left(U_2, U_1\right)\},$$

where $e\left(U_i, U_j\right)$, $i, j = 1, 2$, represents the *excess of* U_i *over* U_j. Recall that (see [1, Lemma 1.5.1] for the last equality)

$$\begin{aligned} e\left(U_i, U_j\right) &:= \inf\left\{\varepsilon > 0 \mid U_i \subset U_j + \varepsilon\mathbb{B}\right\} \\ &= \sup\left\{d\left(x, U_j\right) \mid x \in U_i\right\} \\ &= \sup\left\{d\left(x, U_j\right) - d\left(x, U_i\right) \mid x \in \mathbb{R}^{n+1}\right\}. \end{aligned} \tag{3}$$

Here $\mathbb{B}$ represents the unit open ball in $\mathbb{R}^{n+1}$ endowed with the norm

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| = \max\left\{\|u\|_*, |v|\right\}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{n+1}, \tag{4}$$

where $\|\cdot\|_*$ represents the dual norm in $\mathbb{R}^n$ given by $\|u\|_* = \sup_{\|x\| \leq 1} u'x$.

For any set T, the space of functions $\left(\mathbb{R}^{n+1}\right)^T$ is endowed with the uniform convergence topology, through the (extended) Chebyshev (supremum) distance $d_\infty : \left(\mathbb{R}^{n+1}\right)^T \times \left(\mathbb{R}^{n+1}\right)^T \to [0, +\infty]$, given by

$$d_\infty\left(\sigma_1, \sigma_2\right) := \sup_{t \in T} \left\|\sigma_1(t) - \sigma_2(t)\right\|.$$

From now on, $\mathbb{B}_H(U; \varepsilon)$ and $\mathbb{B}_\infty(\sigma; \varepsilon)$ represent the open balls of radius $\varepsilon > 0$ centered at $U \in CL\left(\mathbb{R}^{n+1}\right)$ and $\sigma \in \left(\mathbb{R}^{n+1}\right)^T$, respectively, with respect to the Hausdorff and Chebyshev distances (for the sake of simplicity, $\mathbb{B}_\infty(\sigma; \varepsilon)$ represents a ball in all spaces $\left(\mathbb{R}^{n+1}\right)^T$, for any T, which will be distinghished by the context).

2.2 Calmness of Multifunctions

Consider a generic multifunction between metric spaces Y and X (with distances denoted indistinctly by d), $\mathcal{M} : Y \rightrightarrows X$. The multifunction $\mathcal{M}$ is said to be *calm* at $(\bar{y}, \bar{x}) \in \text{gph}\mathcal{M}$ if there exist a constant $\kappa \geq 0$ and neighborhoods W of $\bar{x}$ and V of $\bar{y}$ such that

$$d\left(x, \mathcal{M}\left(\bar{y}\right)\right) \leq \kappa d\left(y, \bar{y}\right), \text{ whenever } x \in \mathcal{M}\left(y\right) \cap W \text{ and } y \in V. \tag{5}$$

Sometimes we will be interested in finding some specific neighborhoods and calmness constants; in order to make explicit reference to these elements, we say that $\mathcal{M}$ is *calm at* $(\overline{y}, \overline{x})$ *with constant* κ *on* $V \times W$ when (5) holds.

The calmness property is known to be equivalent to the *metric subregularity* of the inverse multifunction $\mathcal{M}^{-1} : X \rightrightarrows Y$, given by $\mathcal{M}^{-1}(x) := \{y \in Y \mid x \in \mathcal{M}(y)\}$; the metric subregularity of $\mathcal{M}^{-1}$ at $(\overline{x}, \overline{y}) \in \mathrm{gph}\mathcal{M}^{-1}$ is stated in terms of the existence of a (possibly smaller) neighborhood W of $\overline{x}$, as well as a constant $\kappa \geq 0$, such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d\left(\overline{y}, \mathcal{M}^{-1}(x)\right), \text{ for all } x \in W. \tag{6}$$

In other words, (6) can be read as: $\mathcal{M}$ is calm at $(\overline{y}, \overline{x})$ with constant κ on $Y \times W$. The reader is addressed to the monographs [9, 12–14] for a comprehensive analysis of these notions among others variational concepts.

The infimum of all possible constants κ in (5) (for some associated W and V) is equal to the infimum of constants κ in (6) and is called the *calmness modulus* of $\mathcal{M}$ at $(\overline{y}, \overline{x})$, denoted as $\mathrm{clm}\mathcal{M}(\overline{y}, \overline{x})$, defined as ∞ if $\mathcal{M}$ is not calm at $(\overline{y}, \overline{x})$.

3 Calmness via an Indexation Strategy

In this section we discuss three indexation strategies. The first one, the projection strategy $\mathscr{J}$, at a first glance seems to be the most natural, but it turns out not to be adequate as far as

$$d_\infty\left(\mathscr{J}_U, \mathscr{J}_{U_0}\right) \gg d_H(U, U_0) \tag{7}$$

may occur in any neighborhood of a given $U_0 \in CL\left(\mathbb{R}^{n+1}\right)$; where the notation $\gg$ means $\limsup_{U \to U_0}\left(d_\infty\left(\mathscr{J}_U, \mathscr{J}_{U_0}\right)/d_H(U, U_0)\right) = \infty$; see Example 3.1 below. The second strategy, traced out from [6], acts on pairs of closed subsets, say $(U_1, U_2) \mapsto \mathscr{J}_{U_1;U_2}$, and satisfies

$$d_\infty\left(\mathscr{J}_{U_1;U_2}, \mathscr{J}_{U_2;U_1}\right) = d_H(U_1, U_2). \tag{8}$$

The main drawback of this strategy is that, for a given $U_0 \in CL\left(\mathbb{R}^{n+1}\right)$, the indexation of the nominal system U_0 depends on the system U we are comparing with. The third strategy, giving rise to the aimed indexation mapping $\mathscr{I}$, gathers the good features of the other two, as far as it provides an indexation of any system U exclusively in terms of U and the nominal system U_0 and satisfies

$$d_\infty\left(\mathscr{I}_U, \mathscr{I}_{U_0}\right) = d_H(U, U_0) \tag{9}$$

(see Theorem 3.1), which turns out to be enough for the study of the calmness of $\mathscr{F}$ at (U_0, x_0) for any given $x_0 \in \mathscr{F}(U_0)$.

Hereafter in the paper we consider a given nominal set (or system) $U_0 \in CL\left(\mathbb{R}^{n+1}\right)$ and an arbitrarily chosen selection, P, of the *metric projection* multifunction, $\Pi : \mathbb{R}^{n+1} \times CL\left(\mathbb{R}^{n+1}\right) \rightrightarrows \mathbb{R}^{n+1}$, which is given by

$$\Pi\left(t, U\right) := \left\{z \in U \mid \|t - z\| = d\left(t, U\right)\right\}, \quad \left(t, U\right) \in \mathbb{R}^{n+1} \times CL\left(\mathbb{R}^{n+1}\right).$$

Observe that $\Pi\left(t, U\right)$ is always non-empty by the closedness of U. For simplicity we will write $P_U\left(t\right)$ instead of $P\left(t, U\right)$.

3.1 The Projection Strategy

We define $\mathscr{J} : CL\left(\mathbb{R}^{n+1}\right) \to \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$ as $\mathscr{J}_U := P_U$ for all $U \in CL\left(\mathbb{R}^{n+1}\right)$. Now we are going to show an example where (7) happens even for compact convex sets.

Example 3.1 Consider $\mathbb{R}^2$ endowed with the Euclidean norm and let $\mathbb{R}^3$ be equipped with the norm (4).

$$U_0 := \{(x_1, x_2, 0)' \in \mathbb{R}^3 \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le x_1^{2/3}\}$$

and pick any $\varepsilon > 0$. If we move from $\left(\varepsilon, \varepsilon^{2/3}, 0\right)'$ orthogonally to the surface $x_2 = x_1^{2/3}$ until we meet the plane $x_1 = 0$, then we reach $u_\varepsilon := \left(0, \varepsilon^{2/3} + \frac{3}{2}\varepsilon^{4/3}, 0\right)'$. If our orthogonal movement starts at $\left(\frac{8}{27}\varepsilon, \frac{4}{9}\varepsilon^{2/3}, 0\right)'$ and ends at the plane $x_1 = -\varepsilon^{1/3}$, then we reach $z_\varepsilon := \left(-\varepsilon^{1/3}, \frac{13}{9}\varepsilon^{2/3} + \frac{8}{27}\varepsilon^{4/3}, 0\right)'$. For each $\varepsilon > 0$ let

$$U_\varepsilon = \text{conv}\left(U_0 \cup \{u_\varepsilon\}\right).$$

In this case, for $0 < \varepsilon \le (24/65)^{3/2}$ in order to guarantee $\frac{13}{9}\varepsilon^{2/3} + \frac{8}{27}\varepsilon^{4/3} \ge \varepsilon^{2/3} + \frac{3}{2}\varepsilon^{4/3}$, we have

$$P_{U_0}\left(z_\varepsilon\right) = \left(\frac{8}{27}\varepsilon, \frac{4}{9}\varepsilon^{2/3}, 0\right)' \text{ and } P_{U_\varepsilon}\left(z_\varepsilon\right) = u_\varepsilon.$$

Accordingly, as $\varepsilon \downarrow 0$ we have

$$\begin{aligned} d_\infty\left(\mathscr{J}_{U_\varepsilon}, \mathscr{J}_{U_0}\right) &\ge \left\|P_{U_\varepsilon}\left(z_\varepsilon\right) - P_{U_0}\left(z_\varepsilon\right)\right\| \approx \frac{5}{9}\varepsilon^{2/3}, \\ d_H\left(U_\varepsilon, U_0\right) &= \left\|\left(0, \varepsilon^{2/3} + \frac{3}{2}\varepsilon^{4/3}, 0\right)' - \left(\varepsilon, \varepsilon^{2/3}, 0\right)'\right\| \approx \varepsilon, \\ d\left(z_\varepsilon, U_0\right) &\approx \varepsilon^{1/3}, \end{aligned}$$

where $\approx$ means (as usual) that the quotient between left-hand and right-hand sides tends to 1 as $\varepsilon \downarrow 0$. This clearly entails $d_\infty\left(\mathscr{J}_{U_\varepsilon}, \mathscr{J}_{U_0}\right) \gg d_H\left(U_\varepsilon, U_0\right)$, even if $\mathbb{R}^{n+1}$ is replaced with any neighborhood of U_0 (i.e., a set of the form $U_0 + \delta\mathbb{B}$, for any $\delta > 0$).

3.2 The Pairwise Strategy

The following indexation strategy is inspired in [6, Theorem 4.2]. For each $U_1, U_2 \in CL\left(\mathbb{R}^{n+1}\right)$ let us define $\mathscr{J}_{U_1;U_2} \in \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$ given by

$$\mathscr{J}_{U_1;U_2}(t) := \begin{cases} P_{U_1}(t) & \text{if } t \in U_1 \cup U_2, \\ \binom{0_n}{1} & \text{if } t \notin U_1 \cup U_2. \end{cases}$$

Observe that $\binom{0_n}{1}$ is associated with the trivial inequality $0_n' x \leq 1$.

The proof of (8) for this pairwise indexation mapping is essentially given in [6, Theorem 4.2], although in that theorem the uncertainty is confined to the left-hand-side coefficients. Example 3.1 shows that points $t \notin U_1 \cup U_2$ may 'spoil' $d_\infty\left(\mathscr{J}_{U_1}, \mathscr{J}_{U_2}\right)$ in relation to the projection strategy. As said at the beginning of this section, the main drawback of the current pairwise strategy is that the indexation of the nominal system U_0 depends on the system U we are comparing with. In other words, when U varies around a fixed U_0, the indexations of U_0 vary with U, so that we cannot apply the literature background to a fixed $\sigma_0 \in \left(\mathbb{R}^{n+1}\right)^T$. Recalling the indexation mapping $\mathscr{J}$ providing the projection strategy in Sect. 3.1, we immediately observe that $\mathscr{J}_U$ coincides with our current $\mathscr{J}_{U;\mathbb{R}^{n+1}}$.

3.3 The U_0-Based Strategy

Now we are going to define the indexation function $\mathscr{I}$ announced at the beginning of this section. Recall that we are considering a given nominal set $U_0 \in CL\left(\mathbb{R}^{n+1}\right)$, although, for the sake of simplicity, the notation does not reflect the dependence on U_0. We define $\mathscr{I} : CL\left(\mathbb{R}^{n+1}\right) \to \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$ as follows: For each $U \in CL\left(\mathbb{R}^{n+1}\right)$, let $\sigma := \mathscr{I}_U \in \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$ be given by

$$\sigma(t) := \begin{cases} t, & \text{if } t \in U, \\ (P_U \circ P_{U_0})(t), & \text{if } t \notin U. \end{cases} \tag{10}$$

In this way, one easily checks that $\operatorname{rge}\mathscr{I}_U = U$, for all $U \in CL\left(\mathbb{R}^{n+1}\right)$, Obviously, $\mathscr{I}_{U_0} = P_{U_0}$. Next we establish (9).

Theorem 3.1 *Let $\mathscr{I} : CL\left(\mathbb{R}^{n+1}\right) \to \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$ be the indexation function defined in (10) and let $\sigma_0 := \mathscr{I}_{U_0}$. Then,*

$$d_\infty (\sigma, \sigma_0) = d_H (U, U_0), \text{ whenever } \sigma = \mathscr{I}_U, \ U \in CL\left(\mathbb{R}^{n+1}\right). \tag{11}$$

Proof Consider any $U \in CL\left(\mathbb{R}^{n+1}\right)$. In order to establish the inequality '$\leq$' in (11), take any $t \in \mathbb{R}^{n+1}$ and distinguish two cases: If $t \in U$, then

$$\|\sigma (t) - \sigma_0 (t)\| = \left\|t - P_{U_0} (t)\right\| = d (t, U_0) \leq e (U, U_0),$$

where we have taken (3) into account. Otherwise, if $t \notin U$, then

$$\begin{aligned} \|\sigma (t) - \sigma_0 (t)\| &= \left\|(P_U \circ P_{U_0})(t) - P_{U_0} (t)\right\| \\ &= d\left(P_{U_0} (t), U\right) \leq e (U_0, U). \end{aligned}$$

So, $\|\sigma (t) - \sigma_0 (t)\| \leq d_H (U, U_0)$, for all $t \in \mathbb{R}^{n+1}$, and then $d_\infty (\sigma, \sigma_0) \leq d_H (U, U_0)$.

Let us see the opposite inequality. We have that

$$\begin{aligned} e (U, U_0) &= \sup_{t\in U} d (t, U_0) = \sup_{t\in U} d\left(t, P_{U_0} (t)\right) \\ &= \sup_{t\in U} d (\sigma (t), \sigma_0 (t)) \leq d_\infty (\sigma, \sigma_0). \\ e (U_0, U) &= \sup_{t\in U_0} d (t, U) = \sup_{t\in U_0} d (t, P_U (t)) \\ &= \sup_{t\in U_0} d\left(P_{U_0} (t), P_U\left(P_{U_0} (t)\right)\right) \\ &= \sup_{t\in U_0} d (\sigma_0 (t), \sigma (t)) \leq d_\infty (\sigma, \sigma_0). \end{aligned}$$

Consequently, $d_\infty (\sigma, \sigma_0) \geq d_H (U, U_0)$. □

Finally, the following result formalizes the fact that the calmness of $\mathscr{F}^{\mathbb{R}^{n+1}}$ turns out to be equivalent to the calmness of $\mathscr{F}$, with the same constants and closely related neighborhoods.

Theorem 3.2 *Let $x_0 \in \mathscr{F} (U_0)$, $W \subset \mathbb{R}^n$ be a neighborhood of x_0, and $\sigma_0 = \mathscr{I}_{U_0}$. Then $\mathscr{F}^{\mathbb{R}^{n+1}}$ is calm at (σ_0, x_0) with constant $\kappa \geq 0$ on $\mathbb{B}_\infty (\sigma_0; \varepsilon) \times W$ if and only if $\mathscr{F}$ is calm at (U_0, x_0) with the same constant κ on $\mathbb{B}_H (U_0; \varepsilon) \times W$.*

Proof First assume that $\mathscr{F}^{\mathbb{R}^{n+1}}$ is calm at (σ_0, x_0) with constant κ on $\mathbb{B}_\infty (\sigma_0; \varepsilon) \times W$. From Theorem 3.1 we get $\mathscr{I}^{-1} (\mathbb{B}_\infty (\sigma_0; \varepsilon)) = \mathbb{B}_H (U_0; \varepsilon)$. Take any $(U, x) \in \mathbb{B}_H (U_0; \varepsilon) \times W$, such that $x \in \mathscr{F} (U)$ and let $\sigma = \mathscr{I}_U \in \mathbb{B}_\infty (\sigma_0; \varepsilon)$.

Then, applying Theorem 3.1 we have

$$d (x, \mathscr{F} (U_0)) = d\left(x, \mathscr{F}^{\mathbb{R}^{n+1}} (\sigma_0)\right) \leq \kappa d_\infty (\sigma, \sigma_0) = \kappa d_H (U, U_0).$$

On the other hand, assume that $\mathscr{F}$ is calm at (U_0, x_0) with constant κ on $\mathbb{B}_H(U_0;\varepsilon) \times W$. Picking any $\sigma \in \mathbb{B}_\infty(\sigma_0;\varepsilon)$ and defining $U := \mathrm{cl}\sigma\left(\mathbb{R}^{n+1}\right)$, i.e., $U = \mathrm{cl}\{\sigma(t), t \in \mathbb{R}^{n+1}\}$, it is clear from the definitions that $d_\infty(\sigma,\sigma_0) \geq d_H(U,U_0)$. More in detail, for each $t \in \mathbb{R}^{n+1}$ we have $\|\sigma(t) - \sigma_0(t)\| \geq d(\sigma(t), U_0)$ and, accordingly,

$$d_\infty(\sigma,\sigma_0) = \sup_{t\in\mathbb{R}^{n+1}} \|\sigma(t) - \sigma_0(t)\| \geq \sup_{t\in\mathbb{R}^{n+1}} d(\sigma(t), U_0) = e(U, U_0).$$

In a completely analogous way se obtain $d_\infty(\sigma,\sigma_0) \geq e(U_0, U)$. Consequently $d_H(U,U_0) \leq \varepsilon$ and

$$d\left(x, \mathscr{F}^{\mathbb{R}^{n+1}}(\sigma_0)\right) = d(x, \mathscr{F}(U_0)) \leq \kappa d_H(U,U_0) \leq \kappa d_\infty(\sigma,\sigma_0).$$

□

4 Calmness and Minimal Indexations

This section tackles the question of replacing $\mathbb{R}^{n+1}$ with a smaller index set. In fact, keeping the notation of the previous sections, if we consider $U_0 \in CL\left(\mathbb{R}^{n+1}\right)$, and the corresponding indexed system $\sigma_0 = \mathscr{I}_{U_0} \in \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$, we wonder if U_0 itself could play the role of the index set, yielding to a certain minimal indexation (where repetitions of constraints are eliminated).

Let us consider $U_0 \subset T \subset \mathbb{R}^{n+1}$, and the corresponding feasible set mapping, $\mathscr{F}^T : \left(\mathbb{R}^{n+1}\right)^T \rightrightarrows \mathbb{R}^n$ defined in (2). From now on $\sigma_0|_T : T \to \mathbb{R}^{n+1}$ represents the usual rectriction of function σ_0 to the domain T. Obviously

$$\sigma_0|_{U_0}(t) = \sigma_0|_T(t) = \sigma_0(t) = P_{U_0}(t) = t, \text{ for all } t \in U_0.$$

Accordingly, $\mathrm{rge}\ \sigma_0|_{U_0} = \mathrm{rge}\ \sigma_0|_T = \mathrm{rge}\sigma_0 = U_0$, which entails

$$\mathscr{F}^{U_0}\left(\sigma_0|_{U_0}\right) = \mathscr{F}^T(\sigma_0|_T) = \mathscr{F}^{\mathbb{R}^{n+1}}(\sigma_0) = \mathscr{F}(U_0).$$

Roughly speaking, $\sigma_0|_{U_0}$, $\sigma_0|_T$ and σ_0 correspond to three systems with different index sets but having the same coefficient vector set, U_0. So, $\sigma_0|_T$ and σ_0 are formed by the same inequalities as $\sigma_0|_{U_0}$ but with different amount of repetitions. In order to identify the repetitions of constraints in σ_0, we define the following sets of indices:

$$R_{t_0} := \left\{t \in \mathbb{R}^{n+1} \mid \sigma_0(t) = t_0\right\},\ t_0 \in U_0;$$

so, $t \in R_{t_0}$ is indexing an inequality which is a repetition of the one associated with $t_0 \in U_0$. Clearly $\left\{R_{t_0}\right\}_{t_0\in U_0}$ constitutes a partition of $\mathbb{R}^{n+1}$.

Theorem 4.1 *Let $U_0, T \in CL\left(\mathbb{R}^{n+1}\right)$ with $U_0 \subset T$. Let $x_0 \in \mathscr{F}(U_0)$ and $W \subset \mathbb{R}^n$ be a neighborhood of x_0. Let $\sigma_0 = \mathscr{I}_{U_0}$. Then, the following conditions are equivalent:*

(i) $\mathscr{F}^{\mathbb{R}^{n+1}}$ is calm at (σ_0, x_0) with constant κ on $\mathbb{B}_\infty(\sigma_0; \varepsilon) \times W$;
(ii) $\mathscr{F}^T$ is calm at $(\sigma_0|_T, x_0)$ with constant κ on $\mathbb{B}_\infty(\sigma_0|_T; \varepsilon) \times W$;
(iii) $\mathscr{F}^{U_0}$ is calm at $\left(\sigma_0|_{U_0}, x_0\right)$ with constant κ on $\mathbb{B}_\infty\left(\sigma_0|_{U_0}; \varepsilon\right) \times W$.

Moreover, in the case when $U_0 = \operatorname{conv}(T_0)$ with $T_0 \in CL\left(\mathbb{R}^{n+1}\right)$, the following condition is also equivalent to the previous ones:
(iv) $\mathscr{F}^{T_0}$ is calm at $\left(\sigma_0|_{T_0}, x_0\right)$ with constant κ on $\mathbb{B}_\infty\left(\sigma_0|_{T_0}; \varepsilon\right) \times W$.

Proof $(i) \Rightarrow (ii)$. Consider any $(\sigma, x) \in \operatorname{gph}\mathscr{F}^T \cap (\mathbb{B}_\infty(\sigma_0|_T; \varepsilon) \times W)$ and let us see that $d\left(x, \mathscr{F}^T(\sigma_0|_T)\right) \leq \kappa d_\infty(\sigma, \sigma_0|_T)$. Define $\widetilde{\sigma} \in \left(\mathbb{R}^{n+1}\right)^{\mathbb{R}^{n+1}}$ as an extension of σ in the following natural way:

$$\widetilde{\sigma}(t) := \sigma(t_0), \text{ whenever } t \in R_{t_0} \backslash T, \ t_0 \in U_0.$$

Note that, $\widetilde{\sigma}(t)$ is well defined since for each $t \in \mathbb{R}^{n+1}$ there exists a unique $t_0 \in U_0$ such that $t \in R_{t_0}$ (because of the definition of R_{t_0}).

In this way, one easily checks that $\mathscr{F}^T(\sigma) = \mathscr{F}^{\mathbb{R}^{n+1}}(\widetilde{\sigma})$ and $d_\infty(\widetilde{\sigma}, \sigma_0) = d_\infty(\sigma, \sigma_0|_T) < \varepsilon$. In fact, for each $t_0 \in U_0$ and each $\widetilde{t} \in R_{t_0} \backslash T$ we have

$$\begin{aligned}\left\|\widetilde{\sigma}\left(\widetilde{t}\right) - \sigma_0\left(\widetilde{t}\right)\right\| &= \|\sigma(t_0) - \sigma_0(t_0)\| \leq \sup_{t \in U_0} \|\sigma(t) - \sigma_0(t)\| \\ &\leq \sup_{t \in T} \|\sigma(t) - \sigma_0(t)\| = d_\infty(\sigma, \sigma_0|_T).\end{aligned}$$

Accordingly, $d_\infty(\widetilde{\sigma}, \sigma_0) = d_\infty(\sigma, \sigma_0|_T)$. Then, applying (i) we have our aimed inequality

$$d\left(x, \mathscr{F}^T(\sigma_0|_T)\right) = d\left(x, \mathscr{F}^{\mathbb{R}^{n+1}}(\sigma_0)\right) \leq \kappa d_\infty(\widetilde{\sigma}, \sigma_0) = k d_\infty(\sigma, \sigma_0|_T).$$

$(ii) \Rightarrow (iii)$. It is completely analogous to $(i) \Rightarrow (ii)$.

$(iii) \Rightarrow (i)$. Assume (iii), take any $(\sigma, x) \in \operatorname{gph}\mathscr{F}^{\mathbb{R}^{n+1}} \cap (\mathbb{B}_\infty(\sigma_0; \varepsilon) \times W)$, and let us show that

$$d\left(x, \mathscr{F}^{\mathbb{R}^{n+1}}(\sigma_0)\right) \leq \kappa d_\infty(\sigma, \sigma_0).$$

Since $\sigma|_{U_0}$ may be seen as a subsystem of system σ, we immediately have that

$$\mathscr{F}^{\mathbb{R}^{n+1}}(\sigma) \subset \mathscr{F}^{U_0}\left(\sigma|_{U_0}\right) \text{ and } d_\infty\left(\sigma|_{U_0}, \sigma_0|_{U_0}\right) \leq d_\infty(\sigma, \sigma_0) < \varepsilon.$$

So, we have $\left(\sigma|_{U_0}, x\right) \in \operatorname{gph}\mathscr{F}^{U_0} \cap \left(\mathbb{B}_\infty\left(\sigma_0|_{U_0}; \varepsilon\right) \times W\right)$ and

$$d\left(x, \mathscr{F}^{\mathbb{R}^{n+1}}(\sigma_0)\right) = d\left(x, \mathscr{F}^{U_0}\left(\sigma_0|_{U_0}\right)\right) \leq k d_\infty\left(\sigma|_{U_0}, \sigma_0|_{U_0}\right) \leq k d_\infty(\sigma, \sigma_0).$$

From now on we assume that $U_0 = \operatorname{conv}(T_0)$ for some $T_0 \in CL\left(\mathbb{R}^{n+1}\right)$. In this case, we are going to establish $(iii) \Leftrightarrow (iv)$.

$(iii) \Rightarrow (iv)$. Assume that $\mathcal{F}^{U_0}$ is calm at $\left(\sigma_0|_{U_0}, x_0\right)$ with constant κ on $\mathbb{B}_\infty\left(\sigma_0|_{U_0}; \varepsilon\right) \times W$.

Take any $(\sigma, x) \in \operatorname{gph}\mathcal{F}^{T_0} \cap \left(\mathbb{B}_\infty\left(\sigma_0|_{T_0}; \varepsilon\right) \times W\right)$ and let us see that

$$d\left(x, \mathcal{F}^{T_0}\left(\sigma_0|_{T_0}\right)\right) \leq \kappa d_\infty\left(\sigma, \sigma_0|_{T_0}\right).$$

To do that, we are going to define an appropiate extension of $\sigma \in \left(\mathbb{R}^{n+1}\right)^{T_0}$, say $\widetilde{\sigma}$, to the domain U_0. Let $\mathbb{R}_+^{(T_0)}$ denote the set of functions from T_0 to $\mathbb{R}_+$ which are zero except at finitely many elements of T_0. For each $t \in T_0$ we define $\lambda^t = \left(\lambda_{t_0}^t\right)_{t_0 \in T_0}$ by $\lambda_{t_0}^t = 1$ if $t_0 = t$ and $\lambda_{t_0}^t = 0$ otherwise. For each $t \in U_0 \backslash T_0$, recalling $U_0 = \operatorname{conv}(T_0)$, choose arbitrarily $\lambda^t \in \mathbb{R}_+^{(T_0)}$ satisfying $t = \sum_{t_0 \in T_0} \lambda_{t_0}^t t_0$, and $\sum_{t_0 \in T_0} \lambda_{t_0}^t = 1$. Then define

$$\widetilde{\sigma}(t) := \sum_{t_0 \in T_0} \lambda_{t_0}^t \sigma(t_0), \quad \text{for all } t \in U_0.$$

In this way, any inequality in $\widetilde{\sigma}$ is a consequence of σ, and σ is a subsystem of $\widetilde{\sigma}$. Therefore, $\mathcal{F}^{U_0}(\widetilde{\sigma}) = \mathcal{F}^{T_0}(\sigma)$. Moreover

$$\begin{aligned} d_\infty\left(\widetilde{\sigma}, \sigma_0|_{U_0}\right) &= \sup_{t \in U_0}\left\|\widetilde{\sigma}(t) - \sigma_0|_{U_0}(t)\right\| = \sup_{t \in U_0}\left\|\sum_{t_0 \in T_0} \lambda_{t_0}^t \sigma(t_0) - t\right\| \\ &= \sup_{t \in U_0}\left\|\sum_{t_0 \in T_0} \lambda_{t_0}^t (\sigma(t_0) - t_0)\right\| \\ &= \sup_{t \in U_0}\left\|\sum_{t_0 \in T_0} \lambda_{t_0}^t (\sigma(t_0) - \sigma_0|_{T_0}(t_0))\right\| = d_\infty\left(\sigma, \sigma_0|_{T_0}\right). \end{aligned}$$

The last equality comes from the triangular inequality together with the definiton of λ^t for $t \in T_0$. Consequently, $(\widetilde{\sigma}, x) \in \operatorname{gph}\mathcal{F}^{U_0} \cap \left(\mathbb{B}_\infty\left(\sigma_0|_{U_0}; \varepsilon\right) \times W\right)$ and, then,

$$d\left(x, \mathcal{F}^{T_0}\left(\sigma_0|_{T_0}\right)\right) = d\left(x, \mathcal{F}^{U_0}\left(\sigma_0|_{U_0}\right)\right) \leq k d_\infty\left(\widetilde{\sigma}, \sigma_0|_{U_0}\right) = k d_\infty\left(\sigma, \sigma_0|_{T_0}\right).$$

Finally, the proof of $(iv) \Rightarrow (iii)$ follows exactly the same argument as $(iii) \Rightarrow (i)$ just by replacing $\mathbb{R}^{n+1}$ and U_0 with U_0 and T_0, respectively. □

5 Calmness Constants for Polyhedral Uncertainty Sets

Throughout this section we assume that $U_0 := \text{conv}\,(T_0)$, where $\emptyset \neq T_0 \subset \mathbb{R}^{n+1}$ is a finite set, say

$$T_0 := \left\{ \begin{pmatrix} \overline{a}_i \\ \overline{b}_i \end{pmatrix} : i = 1, ..., m \right\},$$

with m standing for the cardinality of T_0 (i.e., there are no repetitions). Obviously, as an index set T_0 can be identified with $\{1, ..., m\}$. Let us denote by $\mathcal{F}_{\overline{a}}^{T_0} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ the feasible set mapping associated with the system

$$\left\{ \overline{a}_i' x \leq b_i, i = 1, ..., m \right\},$$

with $b = (b_i)_{i=1,...,m}$ being the parameter to be perturbed around $\overline{b}$. Theorem 4 in [4] provides a point-based formula (depending exclusively on the nominal data) for $\text{clm}\mathcal{F}_{\overline{a}}^{T_0}\left(\overline{b}, x_0\right)$, with $\left(\overline{b}, x_0\right) \in \text{gph}\mathcal{F}_{\overline{a}}^{T_0}$. Further, [5, Theorem 3] provides a point-based neighborhood $U_{\overline{b}}(x_0)$ such that $\text{clm}\mathcal{F}_{\overline{a}}^{T_0}\left(\overline{b}, x_0\right)$ is indeed a calmness constant for $\mathcal{F}_{\overline{a}}^{T_0}$ at $\left(\overline{b}, x_0\right)$ on $\mathbb{R}^m \times U_{\overline{b}}(x_0)$; see also the comment just after (6). Denoting $\overline{\sigma} = \binom{\overline{a}_i}{\overline{b}_i}_{i=1,...,m} \in \left(\mathbb{R}^{n+1}\right)^m \equiv \left(\mathbb{R}^{n+1}\right)^{T_0}$, Theorem 5.1 below provides a way to construct, from [5, Theorem 3], a calmness constant for $\mathcal{F}^{T_0}$ at $(\overline{\sigma}, x_0)$ on a certain neighborhood of $(\overline{\sigma}, x_0)$, which is also provided by Theorem 5.1.

Theorem 5.1 *Assume that $\kappa \geq 0$ is a calmness constant for $\mathcal{F}_{\overline{a}}^{T_0}$ at $\left(\overline{b}, x_0\right) \in \text{gph}\mathcal{F}_{\overline{a}}^{T_0}$ on $\mathbb{R}^{T_0} \times W$, where W is a neighborhood of x_0. Then, for any given $\varepsilon > 0$ and $\overline{\sigma}$ being defined as above, $\kappa\,(\|x_0\| + 1 + \varepsilon)$ is a calmness constant for $\mathcal{F}^{T_0}$ at $(\overline{\sigma}, x_0)$ on $\left(\mathbb{R}^{n+1}\right)^{T_0} \times (W \cap \mathbb{B}(x_0, \varepsilon))$.*

Proof Lemma 10 in [3] establishes, for our norm choice (4),

$$d\left(\overline{\sigma}, \left(\mathcal{F}^{T_0}\right)^{-1}(x)\right) = \frac{\max_{i\in\{1,...,m\}}\left[\overline{a}_t' x - \overline{b}_i\right]_+}{\|x\| + 1} \text{ for all } x \in \mathbb{R}^n,$$

where $[\alpha]_+ := \max\{\alpha, 0\}$ stands for the positive part of $\alpha \in \mathbb{R}$. Also observe that $\max_{i\in\{1,...,m\}}\left[\overline{a}_t' x - \overline{b}_i\right]_+$ may be written as $d_\infty\left(\overline{b}, \left(\mathcal{F}_{\overline{a}}^{T_0}\right)^{-1}(x)\right)$.

Accordingly, for all $x \in W \cap \mathbb{B}(x_0, \varepsilon)$ we have

$$d\left(x, \mathcal{F}^{T_0}(\overline{\sigma})\right) = d\left(x, \mathcal{F}_{\overline{a}}^{T_0}(\overline{b})\right) \leq \kappa d_\infty\left(\overline{b}, \left(\mathcal{F}_{\overline{a}}^{T_0}\right)^{-1}(x)\right)$$
$$= (\|x\|+1)\,\kappa \frac{\max_{i\in\{1,\ldots,m\}}\left[\overline{a}_t' x - \overline{b}_i\right]_+}{\|x\|+1}$$
$$\leq \kappa\,(\|x_0\|+1+\varepsilon)\, d_\infty\left(\overline{\sigma}, \left(\mathcal{F}^{T_0}\right)^{-1}(x)\right). \qquad \square$$

Remark 5.1 (i) A straightforward combination of Theorems 3.2, 4.1 and 5.1 provides a calmness constant and an asociated neighborhood for $\mathcal{F}$ at (U_0, x_0). For comparative purposes see also [4, Theorem 5] in relation to clm$\mathcal{F}^{T_0}(\overline{\sigma}, x_0)$.

(ii) The previous theorem may be applied in the context when T_0 is our nominal (finite) system and the uncertainty on these coefficient vectors leads to a robust counterpart where each coefficient $\binom{\overline{a}_i}{\overline{b}_i}$, $i = 1, \ldots, m$, may move in a closed box centered at such a point. In this way, in the robust counterpart of T_0 we may replace the union of such boxes with its convex hull, which is a polyhedral set. Also observe that the perturbed uncertainty sets need not to be polyhedral.

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