Polyhedral Risk Measures in Multistage Stochastic Programming

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Mini-Workshop EDF R&D / HU / WIAS, Berlin, March 14, 2007

DFG Research Center MATHEON
Mathematics for key technologies

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Outline

Mean-Risk Stochastic Optimization

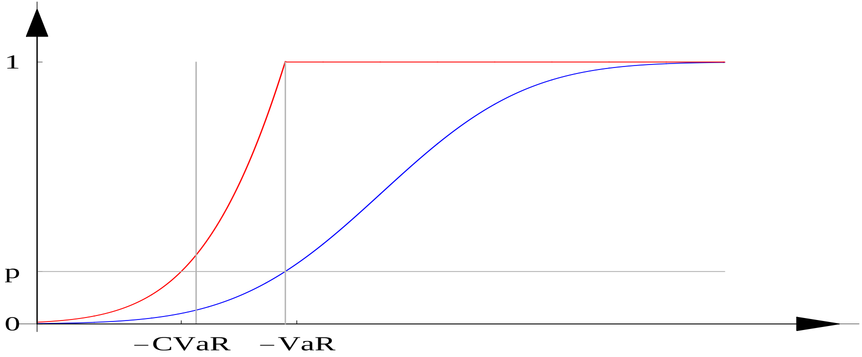
given: $\xi_1 \in \mathbb{R}^d$, $\xi_t \in L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ for $t = 2, \ldots, T$, $\xi^t := (\xi_1, \ldots, \xi_t)$, $\xi := \xi^T$, w.l.o.g. $\mathcal{F} = \sigma(\xi)$

$$\textbf{(SP)} \quad \min \left\{ \begin{array}{l} \gamma \cdot \rho(z_{t_1}, \ldots, z_{t_J}) \\ -(1-\gamma) \cdot \mathbb{E}[z_T] \end{array} \;\middle|\; \begin{array}{l} x_t \in L_{r'}(\Omega, \sigma(\xi^t), \mathbb{P}; \mathbb{R}^{m_t}), \\ x_t \in X_t \text{ a.s.}, \\ A_{t,0}x_t + A_{t,1}(\xi_t)x_{t-1} = h_t(\xi_t) \text{ a.s.} \\ z_t := -\sum_{\tau=1}^{t} \langle b_\tau(\xi_\tau), x_\tau \rangle \\ (t = 1, \ldots, T) \end{array} \right\}$$

- **multistage stochastic program**,
  $\gamma = 0$: classical case, $\gamma \in (0, 1]$: new challenges
- in power applications: big problems
- ρ nonlinear risk functional
  applied to selected timesteps t_j such that $1 \leq t_1 \leq \ldots \leq t_J = T$
- $X_t \subseteq \mathbb{R}^{m_t}$ may contain integrality constraints

Motivation: AVaR

Average Value-at-Risk (Conditional Value-at-Risk) [Rockafellar/Uryasev2002]:



$\mathrm{VaR}_\alpha(z) = -\bar{q}_\alpha(z)$ with $\bar{q}_\alpha(.)$ denoting upper quantile of distribution,
$\mathrm{AVaR}_\alpha(z) := \int_0^\alpha \mathrm{VaR}_{\bar{\alpha}}(z) d\bar{\alpha}$
$= \mathrm{CVaR}_\alpha(z) := \mathbb{E}[\tilde{z}] =$ expectation of tail-distribution

Motivation: AVaR

[Rockafellar/Uryasev2002]:

$$\mathrm{AVaR}_\alpha(z) = \inf\{y_1 + \tfrac{1}{\alpha}\mathbb{E}[(z+y_1)^-] : y_1 \in \mathbb{R}\}$$
$$= \inf\left\{ y_1 + \tfrac{1}{\alpha}\mathbb{E}[y_{2,2}] \;\middle|\; \begin{array}{l} y_1 \in \mathbb{R},\;\; y_2 \in L_1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^2),\;\; y_2 \geq 0 \text{ a.s.}, \\ y_{2,1} - y_{2,2} = y_1 + z \text{ a.s.} \end{array} \right\}$$

consider **(SP)** with $T = 2$, $\gamma = 1$, $\rho(z) = \mathrm{AVaR}_\alpha(z_2) \rightsquigarrow$

$$\min \left\{ y_1 + \tfrac{1}{\alpha}\mathbb{E}[y_{2,2}] \;\middle|\; \begin{array}{l} x_1 \in X_1,\;\; x_2 \in L_{r'}(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^{m_2}) \\ x_2 \in X_2 \text{ a.s. } , A_{t,1}x_1 + A_{t,0}x_2 = h_t \text{ a.s. } , \\ y_1 \in \mathbb{R},\;\; y_2 \in L_1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^2),\;\; y_2 \geq 0 \text{ a.s.} \\ y_{2,1} - y_{2,2} = -\langle b_1, x_1\rangle - \langle b_2, x_2\rangle + y_1 \text{ a.s.} \end{array} \right\}$$

- classical stochastic program with expectation objective
- finite discrete distribution for $\xi_2 \rightsquigarrow$ **linear program** (mixed-integer)

Motivation: Semideviation

Linear Lower Semideviation:

$$\rho(z) = \mathrm{SD}(z) = \mathbb{E}[(z - \mathbb{E}[z])^{-}]$$

Consider **(SP)** with $T = 2$, $\gamma = 1$, $\rho(z) = \mathrm{SD}(z_2)$

$$\min \left\{ \mathbb{E}[y_{2,2}] \;\middle|\; \begin{array}{l} x_1 \in X_1,\; x_2 \in L_{r'}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{m_2}) \\ x_2 \in X_2 \text{ a.s.}, A_{t,1}x_1 + A_{t,0}x_2 = h_t \text{ a.s.}, \\ y_1 \in \mathbb{R}, y_2 \in L_{r'}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2),\; y_2 \geq 0 \text{ a.s.} \\ y_1 = \mathbb{E}[-\langle b_1, x_1\rangle - \langle b_2, x_2\rangle], \\ y_{2,1} - y_{2,2} = -\langle b_1, x_1\rangle - \langle b_2, x_2\rangle - y_1 \text{ a.s.} \end{array} \right\}$$

- finite discrete distribution for $\xi_2 \rightsquigarrow$ **linear program** (mixed-integer)
- "**nasty**" constraint: $y_1 = \mathbb{E}[-\langle b_1, x_1\rangle - \langle b_2(\xi_2), x_2\rangle]$

Multiperiod Risk Measurement

mid-term / long-term activities:
risk measure should consider intermediate wealth values $z_1, \ldots, z_T$

(recall: $z_t := -\sum_{\tau=1}^{t} \langle b_\tau(\xi_\tau), x_\tau \rangle$ with cost coefficients b_τ) in **(SP)**

in order to

- avoid liquidity problems at any time
- anticipate intermediate monitoring by regulation authorities
- reduce the degree of uncertainty at all time

$\rightsquigarrow \quad \rho(z_{t_1}, \ldots, z_{t_J}, \xi)$ instead of just $\rho(z_T)$

Information dynamics $\sigma(\xi_1, \ldots, \xi_t)$, $t = 1, \ldots, T$ may play a role

Definition

Let $z = (z_{t_1}, ..., z_{t_J})$ stochastic wealth process (e.g., $z_t = -\sum_{\tau=1}^{t} \langle b_\tau, x_\tau \rangle$), timesteps $t_0 = 1 < t_1 < ... < t_J = T$, [Eichhorn/Römisch2005]:

$$\rho(z, \xi) = \inf \left\{ \mathbb{E}\left[\sum_{j=0}^{J} \langle c_j, y_j \rangle \right] \middle| \begin{array}{l} y_j \in L_p(\Omega, \sigma(\xi^{t_j}), \mathbb{P}; \mathbb{R}^{k_j}) \ (j = 0, ..., J), \\ y_j \in Y_j \ \text{ a.s. } (j = 0, ..., J), \\ \sum_{\tau=0}^{j} \langle w_{j,\tau}, y_{j-\tau} \rangle = z_{t_j} \ \text{ a.s. } (j = 1, ..., J), \\ \sum_{\tau=0}^{j} V_{j,\tau} y_{j-\tau} = r_j \ \text{ a.s. } (j = 0, ..., J) \end{array} \right\}$$

with dimensions $k_j \in \mathbb{N}$, $d_j \in \mathbb{N}$, vectors $c_j \in \mathbb{R}^{k_j}$, $r_j \in \mathbb{R}^{d_j}$, $w_{j,\tau} \in \mathbb{R}^{k_{j-\tau}}$, matrices $V_{j,\tau} \in \mathbb{R}^{d_j \times k_{j-\tau}}$ $(\tau = 0, ..., j)$, and polyhedral cones $Y_j \subseteq \mathbb{R}^{k_j}$.

- risk = optimal value of a **linear** multistage stochastic program
- wealth values appear at the **right hand sides** of dynamic constraints
- expectation $\mathbb{E}$ only in the objective, simple constraints
- special case $J = 1 \rightsquigarrow$ one-period PRM
- minimization, not maximization

Equivalence of (SP)

why risk as optimal value of a stochastic minimization problem?

consider **(SP)** with $\gamma = 1 \rightsquigarrow$ obvious equivalence to

$$\min \left\{ \mathbb{E}\left[\sum_{j=0}^{J} \langle c_j, y_j \rangle\right] \;\middle|\; \begin{array}{l} x_t \in L_{r'}(\Omega, \sigma(\xi^t), \mathbb{P}; \mathbb{R}^{m_t}),\ x_t \in X_t \text{ a.s. } (t = 1, .., T), \\ y_j \in L_p(\Omega, \sigma(\xi^{t_j}), \mathbb{P}; \mathbb{R}^{k_j}),\ \ y_j \in Y_j \text{ a.s. } (j = 0, ..., J), \\ A_{t,0}x_t + A_{t,1}(\xi_t)x_{t-1} h_t(\xi_t) \text{ a.s. } (t = 2, ..., T) \\ \sum_{\tau=0}^{j} \langle w_{j,\tau}, y_{j-\tau} \rangle = -\sum_{\tau=1}^{t_j} \langle b_\tau(\xi_\tau), x_\tau \rangle \text{ a.s.}, \\ \sum_{\tau=0}^{j} V_{j,\tau} y_{j-\tau} = r_j \text{ a.s. } (j = 1, ..., J) \end{array} \right\}$$

(SP') with $1/p = 1/r + 1/r'$

- classical stochastic program with expectation objective
- well-known solution methods / decomposition approaches
- nonlinearity of risk measure vanished
- additional variables, constraints

Basic Properties

- flexibility through choice of Y_j, $w_{t,\tau}$, c_j, ...
  $\rightsquigarrow$ different attitudes towards risk aversion representable
- a PRM is always **convex**
- if $\rho(z_{t_1}, z_{t_2}, ..., z_{t_J})$ is a PRM then
  $\rho(z_{t_1}, z_{t_2} - z_{t_1}, ..., z_{t_J} - z_{t_{J-1}})$ is a PRM
  $\rho(z_{t_1}, z_{t_2} + z_{t_1}, ..., \sum_{\nu=1}^{J} z_{t_\nu})$ is a PRM

  $\rightsquigarrow$ no matter whether focus is on **income processes** or on **wealth processes** (accumulated),
  $z_{t_j} = -\sum_{t=t_{j-1}+1}^{t_j} \langle b_t(\xi_t), x_t \rangle$ repectively $z_{t_j} = -\sum_{t=1}^{t_j} \langle b_t(\xi_t), x_t \rangle$
- if $\rho(z_{t_1}, z_{t_2}, ..., z_{t_J})$ is a PRM then
  $\rho(z_{t_1}, z_{t_2}, ..., z_{t_J}) + \sum_{j=1}^{J} \mu_j \mathbb{E}[z_{t_j}]$ is a PRM

  $\rightsquigarrow$ **mean-risk models** are fully included

Properties, Dual Representation

Theorem: Assume

- *complete recourse*: $\begin{pmatrix} V_{j,0} \\ w'_{j,0} \end{pmatrix} Y_j = \mathbb{R}^{d_j+1}$ $(j = 1, \dots, J)$,
- *dual feasibility*: $\bigcap_{j=0}^{J} \mathcal{D}_{\rho,j} \neq \emptyset$ with

$$\mathcal{D}_{\rho,j} := \left\{ \begin{array}{l} (u_v, u_w) \in \mathbb{R}^J \times \mathbb{R}^{\sum d_j} : \\ c_j + \sum_{\nu=\max\{1,j\}}^{J} u_{v,\nu} w_{\nu,\nu-j} + \sum_{\nu=j}^{J} V^*_{\nu,\nu-j} u_{w,\nu} \in -Y^*_j \end{array} \right\}$$

Then ρ is **finite, convex, and continuous** with respect to z.
Further, with $1/p + 1/p' = 1$ the functional ρ can be represented by

$$\rho(z, \xi) = \sup \left\{ -\mathbb{E}\left[\sum_{j=1}^{J} \lambda_j z_{t_j} + \langle \mu_j, r_j \rangle \right] \;\middle|\; \begin{array}{l} \lambda_j \in L_{p'}(\Omega, \sigma(\xi^{t_j}), \mathbb{P}), \\ \mu_j \in L_{p'}(\Omega, \sigma(\xi^{t_j}), \mathbb{P}; \mathbb{R}^{d_j}), \\ (\mathbb{E}[\lambda|\xi^{t_j}], \mathbb{E}[\mu|\xi^{t_j}]) \in \mathcal{D}_{\rho,j} \text{ a.s.} \\ (j = 0, \dots, J) \end{array} \right\}$$

This dual representation may serve as criterion for **coherence**.

One-period Instances

- AVaR_α ($J = 1$, see above)
- **Expected utility:** [vonNeumann/Morgenstern1953]

given: utility function $u : \mathbb{R} \to \mathbb{R}$ (concave, increasing),

$$\rho_u(z) := -\mathbb{E}[u(z)]$$

Assume: domain of the revenue z can be bounded a priori.

Then: **piecewise linear utility function** makes sense:

$$u(x) = -\inf\left\{ \langle c, y\rangle \,\middle|\, \begin{array}{l} y \in \mathbb{R}^k, \ y \geq 0 \\ \langle w, y\rangle = x, \ \langle v, y\rangle = 1 \end{array} \right\}$$

according to [Rockafellar/Wets1998], hence,

$$\rho_u(z) = \inf\left\{ \mathbb{E}[\langle c, y\rangle] \,\middle|\, \begin{array}{l} y \in L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^k), \ y \geq 0 \ \text{ a.s.} \\ \langle w, y\rangle = z \ \text{ a.s.}, \ \langle v, y\rangle = 1 \ \text{ a.s.} \end{array} \right\}$$

i.e., ρ_u is a polyhedral risk measure.

Multiperiod Instances

multiperiod extentions of AVaR from [Eichhorn/Römisch2005]:

$$\begin{aligned}
\rho_1(z) &= \tfrac{1}{J}\sum_{j=1}^{J} \mathrm{AVaR}_{\alpha_j}(z_{t_j}) \\
&= \tfrac{1}{J}\sum_{j=1}^{J} \inf_{r\in\mathbb{R}} \left\{ r + \tfrac{1}{\alpha_j}\mathbb{E}\left[\left(z_{t_j}+r\right)^{-} \right] \right\}, \\
\rho_2(z) &= \inf_{r\in\mathbb{R}} \left\{ r + \tfrac{1}{J}\sum_{j=1}^{J} \tfrac{1}{\alpha_j}\mathbb{E}\left[\left(z_{t_j}+r\right)^{-} \right] \right\}.
\end{aligned}$$

- ρ_3: similar to ρ_2 but with consideration of information structure
- ρ_4: multiperiod AVaR extention in the sense of [Riedel2004]
- interpretation not obvious

Multiperiod Instances

No.	primal representation	dual multipliers ($\mu_j = 0$)
ρ_1	$\inf \left\{ \sum_{j=1}^{J} \frac{1}{J} \left(y_{0,j} + \frac{1}{\alpha} \mathbb{E}\left[y_{j,2} \right] \right) \;\middle|\; \begin{array}{l} y_0 \in \mathbb{R}^J \text{ constant}, \\ y_j \in \mathbb{R}_+ \times \mathbb{R}_+ \ \mathcal{F}_{t_j}\text{-meas.} \\ y_{j,1} - y_{j,2} = z_{t_j} + y_{0,j} \\ (j = 1, ..., J) \end{array} \right\}$	$0 \leq \lambda_j \leq \frac{1}{J\alpha}$ $\mathbb{E}\left[\lambda_j\right] = \frac{1}{J} \; (j = 1, ..., J)$
ρ_2	$\inf \left\{ y_0 + \sum_{j=1}^{J} \frac{1}{J\alpha} \mathbb{E}\left[y_{j,2} \right] \;\middle|\; \begin{array}{l} y_0 \in \mathbb{R} \text{ constant}, \\ y_j \in \mathbb{R}_+ \times \mathbb{R}_+ \ \mathcal{F}_{t_j}\text{-measurable} \\ y_{j,1} - y_{j,2} = z_{t_j} + y_0 \\ (j = 1, ..., J) \end{array} \right\}$	$0 \leq \lambda_j \leq \frac{1}{J\alpha}$ $\sum_{j=1}^{J} \mathbb{E}[\lambda_j] = 1$
ρ_3	$\inf \left\{ y_0 + \sum_{j=1}^{J} \frac{1}{J\alpha} \mathbb{E}\left[y_{j,2} \right] \;\middle|\; \begin{array}{l} y_0 \in \mathbb{R} \text{ constant}, \\ y_j \in \mathbb{R}_+ \times \mathbb{R}_+ \ \mathcal{F}_{t_j}\text{-meas.} \\ y_{j,1} - y_{j,2} = z_{t_j} + y_{0,1} + y_{j-1,2} \\ (j = 1, ..., J) \end{array} \right\}$	$\lambda_j + \mathbb{E}[\lambda_{j+1} \mid \mathcal{F}_{t_j}] \leq \frac{1}{J\alpha}$ $0 \leq \lambda_j, \; (j < J)$, $0 \leq \lambda_J \leq \frac{1}{J\alpha}$, $\sum_{j=1}^{J} \mathbb{E}[\lambda_j] = 1$
ρ_4	$\inf \left\{ \frac{1}{J} \left(y_0 + \sum_{j=1}^{J} \frac{1}{\alpha} \mathbb{E}\left[y_{j,2} \right] \right) \;\middle|\; \begin{array}{l} y_0 \in \mathbb{R} \text{ const.}, \\ y_j \in \mathbb{R} \times \mathbb{R}_+ \ \mathcal{F}_{t_j}\text{-meas.} \\ y_J \in \mathbb{R}_+ \times \mathbb{R}_+ \\ y_{j,1} - y_{j,2} = z_{t_j} + y_{t-1,1} \\ (j = 1, ..., J) \end{array} \right\}$	$0 \leq \lambda_t \leq \frac{1}{J\alpha}$, $\lambda_j = \mathbb{E}\left[\lambda_{j+1} \mid \mathcal{F}_{t_j}\right]$ $(j = 1, ..., J-1)$ $\mathbb{E}\left[\lambda_j\right] = \frac{1}{J}$

Value of Perfect Information Based PRM

[Pflug2004], [Pflug/Ruszczyński2005], here: $z = (z_{t_1}, \dots, z_{t_J})$ income process

$$z_{t_j} = -\sum_{t=t_{j-1}+1}^{t_j} \langle b_t(\xi_t), x_t \rangle \quad \text{rather than} \quad z_{t_j} = -\sum_{t=1}^{t_j} \langle b_t(\xi_t), x_t \rangle.$$

negative **utility functional** (with $s_J = 0 \le d \le s_{J-1} \le \dots \le s_1 \le s_0$, $s_{j-1} \le q_j$):

$$\rho_5(z,\xi) = \inf \left\{ \begin{array}{l} -s_0 y_{0,1} + \mathbb{E}\left[\sum_{j=1}^J (-s_j y_{j,1} + q_j y_{j,3}) - d y_{J,2} \right] : \\ y_j \in L_p(\Omega, \sigma(\xi^{t_j}), \mathbb{P}; \mathbb{R}^3) \ (j = 0, \dots, J), \ y_{0,2} = y_{J,1} = 0, \\ y_{j,2} \ge 0 \text{ a.s.}, \ y_{j,3} \ge 0 \text{ a.s. } (j = 1, \dots, J), \\ y_{j,2} - y_{j,3} = y_{j-1,2} + z_{t_j} - y_{j-1,1} \text{ a.s. } (j = 1, \dots, J) \end{array} \right\}$$

$y_{j,1}$: investment/consumption at time t_j, *to be maximized*

$y_{j,2} = (y_{j-1,2} - y_{j-1,1} + z_{t_j})^+$: surplus at time t_j

$y_{j,3} = (y_{j-1,2} - y_{j-1,1} + z_{t_j})^-$: shortfall at time t_j, to be minimized

risk measure value of z given by difference to clairvoyance utility

$$\mathcal{R}(z,\xi) = \rho_5(z,\xi) - \rho_5(z,\mathcal{F}) = \rho_5(z,\xi) + \sum_{j=1}^J s_j \mathbb{E}[z_{t_j}] \ge 0$$

AVaR Applied to Minimum

Obvious multiperiod extention of AVaR:

$$\rho_6(z_{t_1},\dots z_{t_J},\xi) := \mathrm{AVaR}_\alpha(\min\{z_{t_1},\dots z_{t_J}\})$$

can be reformulated as PRM:

$$\rho_6(z,\xi) = \inf\left\{ y_0 + \frac{1}{\alpha}\mathbb{E}[y_{J,2}] \;\middle|\; \begin{array}{l} y_0 \in \mathbb{R},\;\; y_1 \in L_p(\Omega,\sigma(\xi^{t_j}),\mathbb{P};\mathbb{R}^2), \\ y_j \in L_p(\Omega,\sigma(\xi^{t_j}),\mathbb{P};\mathbb{R}^3)\;\;(j=2,\dots,J), \\ y_j \geq 0 \text{ a.s. } \;(j=1,\dots,J), \\ y_{j,1} - y_{j,2} - y_0 = z_{t_j} \text{ a.s. } \;\;(j=1,\dots,J) \\ y_{j,2} - y_{j,3} - y_{j-1,2} = 0 \text{ a.s. } \;\;(j=2,\dots,J) \end{array} \right\}$$

Dual reprentesentation:

$$\rho_6(z,\xi) = \sup\left\{ \mathbb{E}\left[\sum_{j=1}^{J} -\lambda_j z_{t_j}\right] \;\middle|\; \begin{array}{l} \lambda_j \in L_{p'}(\Omega,\sigma(\xi^{t_j}),\mathbb{P}),\;\; \lambda_j \geq 0 \text{ a.s. } (j=1,\dots,J), \\ \mu_j \in L_{p'}(\Omega,\sigma(\xi^{t_j}),\mathbb{P}),\;\; \mu_j \leq 0 \text{ a.s. } (j=2,\dots,J), \\ \sum_{j=1}^{J}\mathbb{E}[\lambda_j] = 1, \\ \lambda_j \leq \mu_j - \mathbb{E}[\mu_{j+1}|\sigma(\xi^{t_j})] \text{ a.s. } (j=1,\dots,J-1), \\ \lambda_J \leq \mu_J + \frac{1}{\alpha} \text{ a.s.} \end{array} \right\}$$

SP with Expectation Objective

Consider **(SP)** with $\gamma = 0$ and the following conditions:

(A1) $\xi = (\xi_1, \ldots, \xi_T) \in L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^s)$

(A2) *relatively complete recourse locally around* ξ:
$\exists \delta_1 > 0 \, \forall \tilde{\xi} \in L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^s)$ with $\|\xi - \tilde{\xi}\|_r < \delta_1$
$\forall x_1 \in X_1, x_\tau \in L_{r'}(\Omega, \sigma(\tilde{\xi}^\tau), \mathbb{P}; \mathbb{R}^{m_\tau})$ with $x_\tau \in \mathcal{X}_\tau(x^{\tau-1}, \tilde{\xi}_\tau)$, $\tau < t$:
$\mathcal{X}_t(x^{t-1}, \tilde{\xi}_t) \neq \emptyset$

(A3) *level-boundedness locally uniformly at* ξ:
$\exists \delta_2 > 0, \varepsilon_0 > 0, B \subset L_{r'}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ bounded
$\forall \tilde{\xi} \in L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^s)$ with $\|\xi - \tilde{\xi}\|_r < \delta_2$: $\quad l_{\varepsilon_0}(F(\tilde{\xi}, .)) \subseteq B$

notations:
$v(\xi)$ optimal value of **(SP)** w.r.t. ξ, $F(\xi, x) = \sum_{t=1}^{T} \mathbb{E}[\langle b_t(\xi_t), x_t \rangle]$ objective function
t-th feasible set $\mathcal{X}_t(x^{t-1}, \xi_t) := \{x_t \in X_t : A_{t,0}x_t + A_{t,1}(\xi_t)x_{t-1} = h_t(\xi_t)\}$
feasible set $\mathcal{X}(\xi) := \{x \in \times_{t=1}^{T} L_{r'}(\Omega, \sigma(\xi^t), \mathbb{P}; \mathbb{R}^{m_t}) : x_1 \in X_1,\ x_t \in \mathcal{X}_t(x^{t-1}, \xi^t) \text{ a.s.}\}$

level set $l_\varepsilon(F(\xi, .)) := \{x \in \mathcal{X}(\xi) : F(\xi, x) \leq v(\xi) + \varepsilon\}$

SP with Expectation Objective

Set r, r' in dependence of the problem class of **(SP)**:

$$r := \begin{cases} \in [1,\infty) & \text{, if only costs or right-hand sides are random} \\ 2 & \text{, if only costs and right-hand sides are random} \\ T & \text{, if all technology matrices are random} \end{cases}$$

$$r' := \begin{cases} \frac{r}{r-1} & \text{, if only costs are random} \\ r & \text{, if only right-hand sides are random} \\ 2 & \text{, if costs and right-hand sides are random} \\ \infty & \text{, if all technology matrices are random} \end{cases}$$

SP with Expectation Objective

Theorem [Heitsch/Römisch/Strugarek06]:

Let (A1)-(A3) be satisfied for **(SP)** with $\gamma = 0$ and let X_1 be bounded. $\Rightarrow$ $\exists \delta, L > 0 \; \forall \tilde{\xi}$ with $\|\xi - \tilde{\xi}\|_r \leq \delta$ it holds for the optimal values that

$$\left| v(\xi) - v(\tilde{\xi}) \right| \leq L \Big(\|\xi - \tilde{\xi}\|_r + D_{\mathrm{f}}(\xi, \tilde{\xi}) \Big)$$

with $D_{\mathrm{f}}(\xi, \tilde{\xi})$ denoting the *filtration distance* given by

$$D_{\mathrm{f}}(\xi, \tilde{\xi}) := \sup_{\varepsilon \in (0, \varepsilon_0]} D_{\mathrm{f},\varepsilon}(\xi, \tilde{\xi})$$

$$D_{\mathrm{f},\varepsilon}(\xi, \tilde{\xi}) := \inf \left\{ \sum_{t=2}^{T-1} \max \left\{ \|\mathbb{E}[x_t | \sigma(\tilde{\xi}^t)] - x_t\|_{r'}, \|\mathbb{E}[\tilde{x}_t | \sigma(\xi^t)] - \tilde{x}_t\|_{r'} \right\} \;\middle|\; \begin{array}{l} x \in l_\varepsilon(F(\xi, \cdot)), \\ \tilde{x} \in l_\varepsilon(F(\tilde{\xi}, \cdot)) \end{array} \right\}$$

- local **calmness** property
- D_{f} depends on problem data (solution behavior), not a metric
- level sets bounded due to (A3) (e.g., if X_t bounded in $\mathbb{R}^{m_t}$)

Stability of SP with PRM Objective

How to generalize this Theorem from $\gamma = 0$ to $\gamma > 0$?

two possibilities:

- try to generalize above theorem to **(SP')**
  $\rightsquigarrow$ problems:
  - different integrability numbers p and r'
  - new recourse matrices $A_{t,0}$ are stochastic $\rightsquigarrow$ technical problems
- **consecutive approach:**
  - first analyze stability of ρ (minimization problem)
  - then analyze **(SP)** with risk objective ($\gamma = 1$)

$\rightsquigarrow$ turns out to be more fruitful

Continuity of PRM

Theorem [Eichhorn/Römisch06]:

Let ρ be a polyhedral risk measure satisfying complete recourse and dual feasibility. Then $\exists K_\rho > 0 \; \forall (z,\xi),(\tilde{z},\tilde{\xi}) \in \mathcal{Z}_\Xi$ it holds that

$$\left|\rho(z,\xi) - \rho(\tilde{z},\tilde{\xi})\right| \leq K_\rho\Big(\|z-\tilde{z}\|_p + D_\rho\big((z,\xi),(\tilde{z},\tilde{\xi})\big)\Big)$$

global **Lipschitz** property !

notations:

$\mathcal{Z}_\Xi := \Big\{(z,\xi) : \xi \in L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^s),\; z \in \times_{j=1}^J L_p(\Omega,\sigma(\xi^{t_j}),\mathbb{P})\Big\}$

filtration distance $\quad D_\rho((z,\xi),(\tilde{z},\tilde{\xi})) := \sup_{\varepsilon>0} D_{\rho,\varepsilon}((z,\xi),(\tilde{z},\tilde{\xi}))$

$$D_{\rho,\varepsilon}((z,\xi),(\tilde{z},\tilde{\xi})) := \inf\left\{\sum_{j=1}^{J-1} \max\left\{\|\bar{y}_j - \mathbb{E}[\bar{y}_j|\sigma(\tilde{\xi}^{t_j})]\|_p, \|\tilde{y}_j - \mathbb{E}[\tilde{y}_j|\sigma(\xi^{t_j})]\|_p\right\} \;\middle|\; \begin{array}{l}\bar{y} \in l_{\rho,\varepsilon}(z,\xi),\\ \tilde{y} \in l_{\rho,\varepsilon}(\tilde{z},\tilde{\xi})\end{array}\right\}$$

level set $l_{\rho,\varepsilon}(z,\xi) = \{y \in \mathcal{Y}(z,\xi) : \sum_{j=0}^J \mathbb{E}[\langle c_j, y_j\rangle] \leq \rho(z,\xi)+\varepsilon\}$ possibly unbounded

Stability SP with PRM Objective

Consider **(SP)** with $\gamma = 1$ and the conditions (A1)-(A3).
Choose $p \in [1, \infty)$ arbitrarily.
Set r, r' in dependence of the problem class of **(SP)**:

$$r := \begin{cases} \in [p, \infty) & , \text{ if only costs or right-hand sides are random} \\ 2p & , \text{ if only costs and right-hand sides are random} \\ pT & , \text{ if all technology matrices are random} \end{cases}$$

$$r' := \begin{cases} \frac{pr}{r-p} & , \text{ if only costs are random} \\ r & , \text{ if only right-hand sides are random} \\ 2p & , \text{ if costs and right-hand sides are random} \\ \infty & , \text{ if all technology matrices are random} \end{cases}$$

$\rightsquigarrow$ $r \geq p$ and $r' \geq p$.

Stability SP with PRM Objective

Theorem [Eichhorn/Römisch06]:

Let (A1)-(A3) be satisfied for **(SP)** with $\gamma = 1$ and let X_1 be bounded. $\Rightarrow$ $\exists \delta, K > 0 \ \forall \tilde{\xi}$ with $\|\xi - \tilde{\xi}\|_r \leq \delta$ it holds for the optimal values that

$$\left| v(\xi) - v(\tilde{\xi}) \right| \leq K \left(\|\xi - \tilde{\xi}\|_r + D_{\mathrm{f},\rho}(\xi, \tilde{\xi}) \right)$$

with $D_{\mathrm{f},\rho}(\xi, \tilde{\xi})$ denoting the *filtration distance w.r.t.* ρ given by

$$D_{\mathrm{f},\rho}(\xi,\tilde{\xi}) := \sup_{\varepsilon \in (0,\varepsilon_0]} D_{\mathrm{f},\rho,\varepsilon}(\xi,\tilde{\xi})$$

$$D_{\mathrm{f},\rho,\varepsilon}(\xi,\tilde{\xi}) :=$$

$$\inf \left\{ \begin{array}{l} \sum_{t=2}^{T-1} \max \left\{ \|\mathbb{E}[x_t | \sigma(\tilde{\xi}^t)] - x_t\|_{r'}, \|\mathbb{E}[\tilde{x}_t | \sigma(\xi^t)] - \tilde{x}_t\|_{r'} \right\} \\ + \sum_{j=1}^{J-1} \max \left\{ \|\mathbb{E}[y_j | \sigma(\tilde{\xi}^{t_j})] - y_j\|_p, \|\mathbb{E}[\tilde{y}_j | \sigma(\xi^{t_j})] - \tilde{y}_j\|_p \right\} \end{array} \middle| \begin{array}{l} x \in l_\varepsilon(F(\xi,\cdot)), \\ \tilde{x} \in l_\varepsilon(F(\tilde{\xi},\cdot)), \\ y \in l_{\rho,\varepsilon}(z(\xi,x),\xi), \\ \tilde{y} \in l_{\rho,\varepsilon}(z(\tilde{\xi},\tilde{x}),\tilde{\xi}) \end{array} \right\}$$

- local calmness, $D_{\mathrm{f},\rho}$ depends on solution behavior, not a metric
- x level sets bounded due to (A3), y level sets may be unbounded

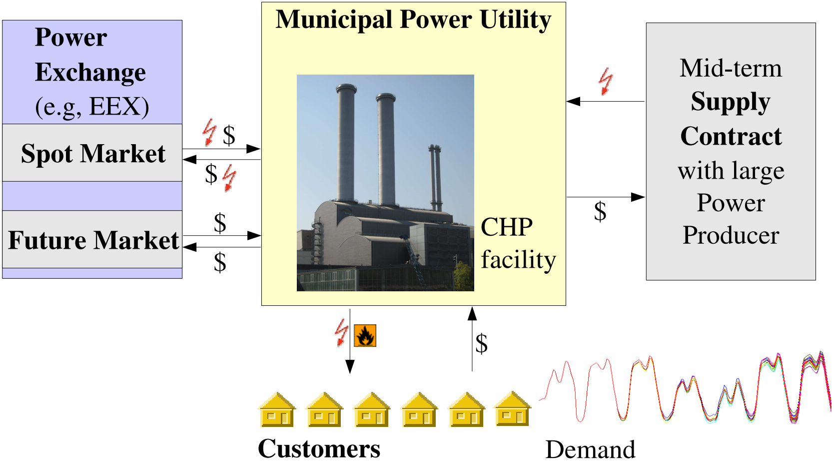
Remarks

- Apparently more degrees of freedom for r, r' since p can be chosen arbitrarily
- For practical use of theorem (**Scenario Tree Approximation**), $D_{\mathrm{f},\rho}$ must be estimated by problem independent objects
- If (A3) and the PRM level sets are uniformly bounded then $\exists C > 0$ such that

$$D_{\mathrm{f},\rho}(\xi,\tilde{\xi}) \leq C \cdot \sup_{\|x\|_{r'}\leq 1} \sum_{t=2}^{T-1} \left\| \mathbb{E}[x_t|\sigma(\xi^t)] - \mathbb{E}[x_t|\sigma(\tilde{\xi}^t)] \right\|_{r'}$$

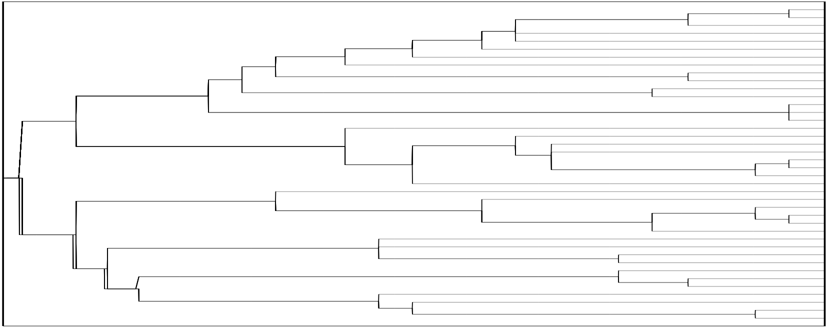
- [Eichhorn/Römisch2007]:
  $p > 1 \rightsquigarrow$ PRM level sets are typically unbounded
  $p = 1 \rightsquigarrow$ level sets of most PRM are uniformly bounded
- **Same scenario tree construction methods** for PRM with $p = 1$! $\rightsquigarrow$ [Heitsch/Römisch2007]

Application



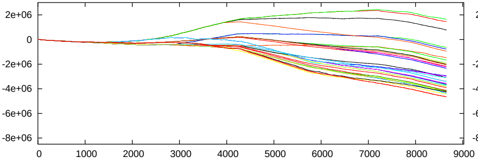
LP model, customer electricity demand exceeds capacity of CHP facility

Application

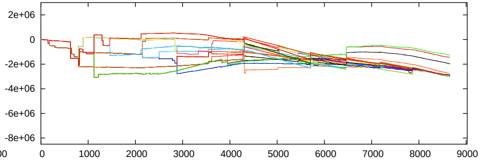


- multivariate scenario tree: electr./heat demand, spot/future prices
- time horizon: one year, hourly discretization, $T = 8760$
- risk measures applied to end of each week, i.e., $J = 52$, $t_j = j \cdot 7 \cdot 24$
- objective: $\min 0.9 \cdot \rho(z_{t_1}, \ldots, z_{t_J}) - 0.1 \cdot \mathbb{E}[z_{t_J}]$

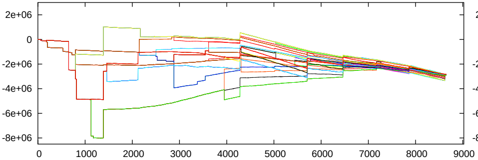
Cash Value Curves (low fuel cost)



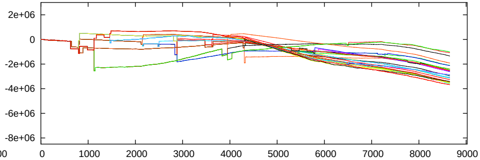
Optimizing $\mathbb{E}[z_T]$ only



$\min 0.9 \cdot \rho_2(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$



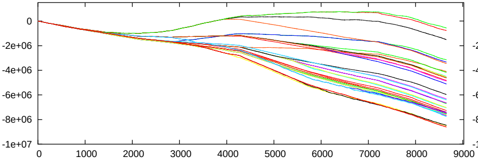
$\min 0.9 \cdot AVaR(z_T) - 0.1 \cdot \mathbb{E}[z_T]$



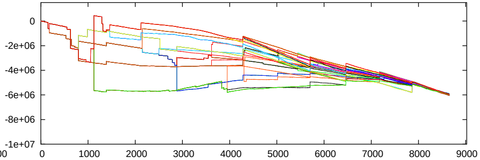
$\min 0.9 \cdot \rho_4(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$

$\rightsquigarrow$ **Fewer spreading with multiperiod risk measures**

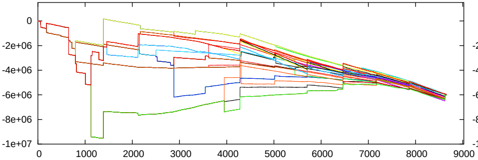
Cash Value Curves



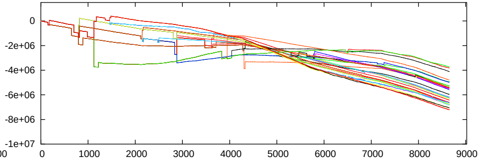
Optimizing $\mathbb{E}[z_T]$ only



$\min 0.9 \cdot \rho_2(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$



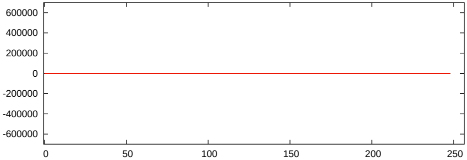
$\min 0.9 \cdot AVaR(z_T) - 0.1 \cdot \mathbb{E}[z_T]$



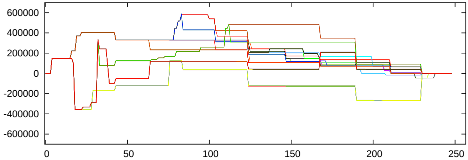
$\min 0.9 \cdot \rho_4(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$

ρ_2: optimal level, ρ_4: spreading equally distributed

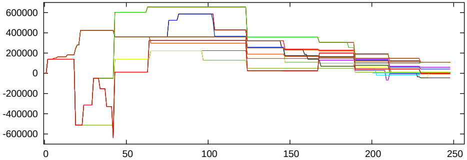
Future Stock



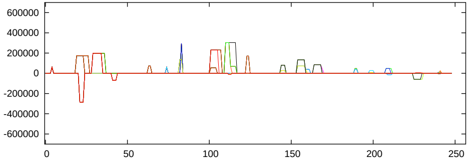
max $\mathbb{E}[z_T]$ only



$\min 0.9 \cdot \rho_2(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$



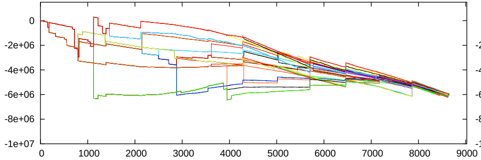
$\min 0.9 \cdot AVaR(z_T) - 0.1 \cdot \mathbb{E}[z_T]$



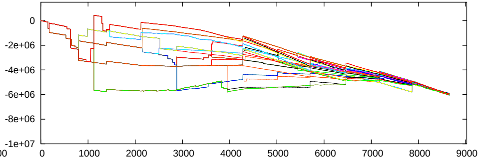
$\min 0.9 \cdot \rho_4(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$

$\rightsquigarrow$ **different future trading strategies**

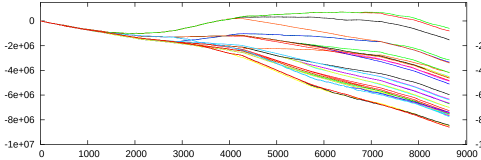
Cash Value Curves (cont.)



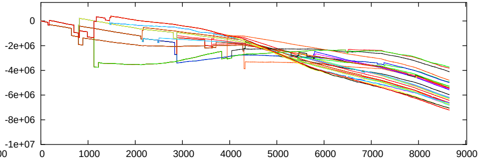
$\min 0.9 \cdot \rho_6(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$



$\min 0.9 \cdot \rho_2(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$



$\min 0.9 \cdot \rho_5(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$

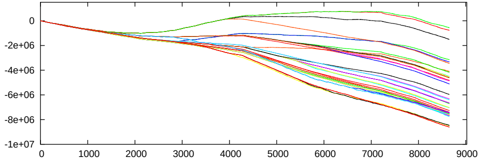


$\min 0.9 \cdot \rho_4(z_1, \ldots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$

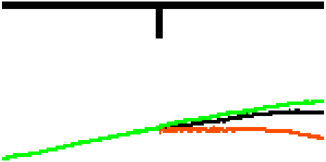
$\rho_6(z_1, \ldots, z_T) = \mathrm{AVaR}(\min_j z_{t_j})$ similar to ρ_2.

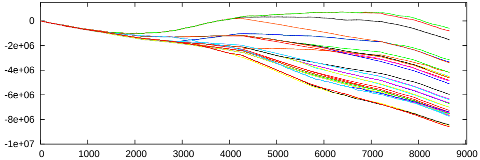
ρ_5 similar to $\mathbb{E}[z_T]$?

Cash Value Curves (cont.)

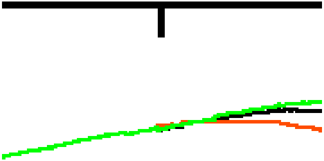


$\max \mathbb{E}[z_T]$





$\min 0.9 \cdot \rho_5(z_1, \dots, z_T) - 0.1 \cdot \mathbb{E}[z_T]$



ρ_5 measures spread at t_j seen from t_{j-1} respectively.

Conclusion

- Polyhedral Risk Measures (PRM) as optimal values of certain simple stochastic minimization problems
- Allow different preferences / strategies for risk aversion
- Problems remain tractable
- Stability Theorem for PRM objective in **(SP)** similar to the case of expectation objective (with slightly different filtration distance)
- PRM instances from [Eichhorn/Römisch2005], [Pflug/Ruszczyński2005]: uniformly bounded level sets, $\rightsquigarrow$ scenario tree construction methods from [Heitsch/Römisch2007] can be used