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Critical Objective Size and Calmness Modulus in Linear Programming

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Abstract This paper introduces the concept of critical objective size associated with a linear program in order to provide operative point-based formulas (only involving the nominal data, and not data in a neighborhood) for computing or estimating the calmness modulus of the optimal set (argmin) mapping under uniqueness of nominal optimal solution and perturbations of all coefficients. Our starting point is an upper bound on this modulus given in Cánovas et al. (2015). In this paper we prove that this upper bound is attained if and only if the norm of the objective function coefficient vector is less than or equal to the critical objective size. This concept also allows us to obtain operative lower bounds on the calmness modulus. We analyze in detail an illustrative example in order to explore some strategies that can improve the referred upper and lower bounds.

Keywords Variational analysis · Calmness · Linear programming

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1 Introduction

Calmness property of multifunctions in relation to optimization has become an active research area of increasing interest due to its repercussions in both theory and algorithms. This paper tries to contribute to this subject in the paradigmatic framework of ordinary linear programming under *full* perturbations by providing exact formulas or tight estimations of the calmness modulus of the argmin mapping when the nominal optimal solution is unique. We emphasize the operativeness of the given expressions or estimations as far as they depend exclusively on the nominal data, not involving data in a neighborhood.

We consider the parametrized linear optimization problem

$$\begin{array}{lll} P(c,a,b): & \text{minimize} & c'x \\ & \text{subject to} & a_t'x \leq b_t, \quad t \in T := \{1,2,...,m\}, \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, and $c \in \mathbb{R}^n$, $a \equiv (a_t)_{t\in T} \in (\mathbb{R}^n)^T$, and $b \equiv (b_t)_{t\in T} \in \mathbb{R}^T$ are the problem's data. All elements in $\mathbb{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \mathbb{R}^n$.

Associated with the previous parametrized problem, we consider the *optimal set mapping*, $\mathcal{S} : \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{S}(c,a,b) := \left\{x \in \mathbb{R}^n \mid x \text{ is an optimal solution of } P(c,a,b)\right\}.$$

The parameter space $\mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$ is endowed with the norm

$$\|(c,a,b)\| := \max\{\|c\|_*, \|(a,b)\|_\infty\}, \tag{2}$$

where $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm* given by $\|u\|_* = \max_{\|x\|\leq 1} |u'x|$, and $\|(a,b)\|_\infty := \max_{t\in T} \left\|\binom{a_t}{b_t}\right\|$, where

$$\left\|\begin{pmatrix} a_t \\ b_t \end{pmatrix}\right\| := \max\{\|a_t\|_*, |b_t|\}. \tag{3}$$

For the sake of simplicity, from now on we abbreviate our nominal parameter as $\overline{p}$; i.e.,

$$\overline{p} := \left(\overline{c}, \overline{a}, \overline{b}\right) \in \mathbb{R}^n \times \left(\mathbb{R}^n\right)^T \times \mathbb{R}^T.$$

The *Slater constraint qualification* (SCQ) is said to hold at parameter $\left(\overline{a}, \overline{b}\right) \in (\mathbb{R}^n)^T \times \mathbb{R}^T$ if there exists $\widehat{x} \in \mathbb{R}^n$ (called a Slater point) such that $\overline{a}_t'\widehat{x} < \overline{b}_t$ for all $t \in T$.

Assumptions From now on, we consider a given $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$ and assume

- $\mathcal{S}(\overline{p}) = \{\overline{x}\}$,
- The SCQ holds at $\left(\overline{a}, \overline{b}\right)$.

(Observe that these assumptions easily imply $\overline{c} \neq 0_n$, where 0_n denotes the zero-vector of $\mathbb{R}^n$.)

The starting point of this work is an upper bound on the calmness modulus of $\mathcal{S}$ provided in [4, Theorem 4.2(i)] under uniqueness of nominal optimal solution. Following the goal of computing the exact calmness modulus of $\mathcal{S}$, we provide a quite tight lower bound, in the sense that it coincides with the upper bound (and then, provides the exact modulus) in a variety of situations. At this moment, we advance that one of these situations is characterized in terms of the size (norm) of vector $\overline{c}$ in the objective function.

Recall that a generic multifunction $\mathcal{M} : Y \rightrightarrows X$ between metric spaces (with distances denoted indistinctly by d) is said to be *calm* at $(\overline{y}, \overline{x}) \in \text{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\left(x, \mathcal{M}\left(\overline{y}\right)\right) \leq \kappa d\left(y, \overline{y}\right) \tag{4}$$

whenever $x \in \mathcal{M}(y) \cap U$ and $y \in V$; where, as usual, $d(x, \Omega)$ is defined as $\inf\{d(x,z) \mid z \in \Omega\}$ for $\Omega \subset \mathbb{R}^n$. It is well-known that the calmness of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ is equivalent to the *metric subregularity* of $\mathcal{M}^{-1}$ at $(\overline{x}, \overline{y})$ (see, for instance, [8, Theorem 3H.3 and Exercise 3H.4]). Recall that $\mathcal{M}^{-1}$ (given by $y \in \mathcal{M}^{-1}(x) \Leftrightarrow x \in \mathcal{M}(y)$) is *metrically subregular* at $(\overline{x}, \overline{y})$ if there exist a constant $\kappa \geq 0$ and a (possibly smaller) neighborhood U of $\overline{x}$ such that

$$d\left(x, \mathcal{M}\left(\overline{y}\right)\right) \leq \kappa d\left(\overline{y}, \mathcal{M}^{-1}(x)\right), \text{ for all } x \in U. \tag{5}$$

The infimum of those $\kappa \geq 0$ for which (4) –or (5)– holds (for some associated neighborhoods) is called the *calmness modulus* of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ and it is denoted by $\text{clm}\mathcal{M}(\overline{y}, \overline{x})$.

More details about this and other variational properties can be traced out from the monographs [8, 14, 18, 21]; see also [9, 12, 15, 16] in relation to the calmness of constraint systems in the context of canonical perturbations; where the calmness property is closely connected with *local error bounds*. Other subdifferential approaches to calmness/local error bounds can be found in [1, 11, 13, 17].

The structure of the paper is as follows. Section 2 provides the necessary notation and preliminary results. Section 3 sharpens the referred [4, Theorem 4.2(i)] by showing that this result can be confined to those KKT index sets (see Section 2) which are minimal with respect to the inclusion order. Section 4 is devoted to obtain a lower bound on the calmness modulus of the argmin mapping $\mathcal{S}$ which leads to the exact modulus when the objective function coefficient vector is small enough. In Section 5 we introduce the so-called critical objective size, providing the threshold for $\|\overline{c}\|$ under which an upper bound existing in the literature becomes the exact calmness modulus. Section 6 is devoted to illustrate by means of examples some strategies providing tighter estimations on the modulus. We finish the paper with a section of conclusions.

2 Preliminaries

In this section we introduce some additional notation and preliminary results which are needed later on. Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $\text{conv}X$ and $\text{cone}X$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that $\text{cone}X$ always contains 0_k, in particular $\text{cone}(\emptyset) = \{0_k\}$. If X is a subset of any topological space, $\text{int}X$, $\text{cl}X$ and $\text{bd}X$ stand, respectively, for the interior, the closure, and the boundary of X.

We denote by $\mathcal{F} : (\mathbb{R}^n)^T \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$ the *feasible set mapping* associated with problem (1), which is given by

$$\mathcal{F}(a,b) := \left\{x \in \mathbb{R}^n \mid a_t'x \leq b_t,\ \ t \in T\right\},\ \ (a,b) \in \left(\mathbb{R}^n\right)^T \times \mathbb{R}^T.$$

In the case of finite linear systems (i.e., when T is finite), it is well-known that, for a given $a \equiv (a_t)_{t \in T}$, $\mathcal{F}(a, \cdot)$ is always calm at any point of its (polyhedral) graph as a consequence of a classical result by Robinson [20]. In the context of canonical perturbations (where perturbations fall on (c, b)), the same result ensures that mapping $\mathcal{S}(\cdot, a, \cdot)$ is always calm at any point of its graph, since the KKT conditions allow us to express the graph of $\mathcal{S}(\cdot, a, \cdot)$ as a finite union of polyhedral sets. This is no longer the case in the current framework of perturbations of all data (i.e., when a becomes a parameter subject to perturbations). In relation to this last framework, [7, Theorem 5] asserts that $\operatorname{clm}\mathcal{F}\left(\left(\overline{a},\overline{b}\right),\overline{x}\right)=(\|\overline{x}\|+1)\operatorname{clm}\mathcal{F}(\overline{a},\cdot)\left(\overline{b},\overline{x}\right)$ at any $\left(\left(\overline{a},\overline{b}\right),\overline{x}\right)\in\operatorname{gph}\mathcal{F}$ and, accordingly, $\mathcal{F}$ is always calm at any point of its graph. Again in the context of perturbations of all data, assuming $\mathcal{S}(\overline{p})=\{\overline{x}\}$ and combining [19, Theorems 1 and 2], [4, Theorem 4.1] establishes a characterization for the calmness of the corresponding argmin mapping $\mathcal{S}$ in the following terms: either the SCQ holds at $\left(\overline{a},\overline{b}\right)$ or $\mathcal{F}\left(\overline{a},\overline{b}\right)$ is a singleton. In the next paragraphs we detail the necessary background about calmness moduli.

Throughout the paper, we appeal to the *set of active indices* at $x\in\mathcal{F}(a,b)$, denoted by $T_{a,b}(x)$ and defined as

$$T_{a,b}(x):=\left\{t\in T\mid a_t'x=b_t\right\}.$$

Associated with a given $(\overline{p},\overline{x})\in\operatorname{gph}\mathcal{S}$, we appeal to the following family of index subsets associated with the Karush-Kuhn-Tucker (KKT) conditions (hereafter referred to as *KKT index sets*)

$$\mathcal{K}_{\overline{p}}(\overline{x})=\left\{D\subset T_{\overline{a},\overline{b}}(\overline{x})\,\middle|\,|D|\leq n\text{ and }-\overline{c}\in\operatorname{cone}\{\overline{a}_t,\ t\in D\}\right\},$$

where $|D|$ stands for the cardinality of D and condition $|D|\leq n$ comes from Caratheodory's Theorem. For any $D\in\mathcal{K}_{\overline{p}}(\overline{x})$ we consider the mapping $\mathcal{L}_D:(\mathbb{R}^n)^T\times\mathbb{R}^T\times(\mathbb{R}^n)^D\times\mathbb{R}^D\rightrightarrows\mathbb{R}^n$ given by

$$\mathcal{L}_D(a,b,u,d):=\left\{x\in\mathbb{R}^n\mid a_t'x\leq b_t,\ t\in T;\ u_t'x\leq d_t,\ t\in D\right\}.\tag{6}$$

Here, analogously to (3), we consider the norm

$$\|(a,b,u,d)\|_\infty:=\max\left\{\|a_t\|_*,|b_t|,\|u_s\|_*,|d_s|:t\in T,\ s\in D\right\}\tag{7}$$

Observe that $\mathcal{L}_D$ is the feasible set mapping associated with an enlarged system with $|D|$ new constraints, so that the existing theory for feasible set mappings may be applied to $\mathcal{L}_D$. Also note that, for $D\in\mathcal{K}_{\overline{p}}(\overline{x})$ and using the notation $\overline{a}_D=(\overline{a}_t)_{t\in D}$, $\overline{b}_D=\left(\overline{b}_t\right)_{t\in D}$, the set $\mathcal{L}_D\left(\overline{a},\overline{b},-\overline{a}_D,-\overline{b}_D\right)$ is nothing else but the set of KKT points of $P\left(\overline{c},\overline{a},\overline{b}\right)$ having D as the KKT index set.

For each $D\in\mathcal{K}_{\overline{p}}(\overline{x})$ let us consider the convex function $f_D:\mathbb{R}^n\longrightarrow\mathbb{R}$ given by

$$\begin{aligned}f_D(x)&:=\max\left\{\overline{a}_t'x-\overline{b}_t,\ t\in T;\ -\overline{a}_t'x+\overline{b}_t,\ t\in D\right\}\\&=\max\left\{\overline{a}_t'x-\overline{b}_t,\ t\in T\setminus D;\ \left|\overline{a}_t'x-\overline{b}_t\right|,\ t\in D\right\}.\end{aligned}$$

The next proposition comes straightforwardly from [2, Lemma 10].

Proposition 2.1 *For any $D\in\mathcal{K}_{\overline{p}}(\overline{x})$ and any $x\in\mathbb{R}^n$ one has*

$$d\left(\left(\overline{a},\overline{b},-\overline{a}_D,-\overline{b}_D\right),\mathcal{L}_D^{-1}(x)\right)=\frac{f_D(x)}{\|x\|+1},$$

where d stands for the distance associated with the norm (7) considered in $(\mathbb{R}^n)^T\times\mathbb{R}^T\times(\mathbb{R}^n)^D\times\mathbb{R}^D$.

Remark 2.1 Recalling the definition of calmness modulus (see (5)), for any $D\in\mathcal{K}_{\overline{p}}(\overline{x})$ the previous proposition clearly entails

$$\operatorname{clm}\mathcal{L}_D\left(\left(\overline{a},\overline{b},-\overline{a}_D,-\overline{b}_D\right),\overline{x}\right)=\limsup_{x\to\overline{x},\ x\neq\overline{x}}\frac{\|x-\overline{x}\|}{f_D(x)/(\|x\|+1)},$$

taking into account that the assumption $\mathcal{S}(\overline{p})=\{\overline{x}\}$ entails $\mathcal{L}_D\left(\overline{a},\overline{b},-\overline{a}_D,-\overline{b}_D\right)=\{\overline{x}\}$ and, accordingly, $f_D(x)>0$ for $x\neq\overline{x}$.

The following result follows by combining [7, Theorem 5] and [3, Proposition 4.1] and provides a more tractable expression for $\mathrm{clm}\mathcal{L}_D\left(\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right), \overline{x}\right)$, as far as it only depends on the nominal data $\overline{p}$ and $\overline{x}$. Here we use the notation

$$C_D := \mathrm{conv}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\,;\, -\overline{a}_t,\ t \in D\} \text{ for } D \in \mathcal{K}_{\overline{p}}(\overline{x})\,.$$

Recall that the assumption $\mathcal{S}(\overline{p}) = \{\overline{x}\}$ entails $-\overline{c} \in \mathrm{int\,cone}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\}$ and, accordingly, $0_n \in \mathrm{int}\, C_D$ for all $D \in \mathcal{K}_{\overline{p}}(\overline{x})$.

Proposition 2.2 *Under the current assumptions, for any $D \in \mathcal{K}_{\overline{p}}(\overline{x})$ we have*

$$\mathrm{clm}\mathcal{L}_D\left(\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right), \overline{x}\right) = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \mathrm{bd}\, C_D\right)},$$

where d_* stands for the distance associated with $\|\cdot\|_*$ in $\mathbb{R}^n$.

The next result constitutes a key tool in the present paper.

Theorem 2.1 [4, Theorem 4.2 (i)] *Under the current assumptions we have*

$$\mathrm{clm}\mathcal{S}(\overline{p}, \overline{x}) \le \max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \mathrm{bd}\, C_D\right)}. \tag{8}$$

3 Minimal KKT Index Sets

The aim of this section is to establish the following refinement of [4, Theorem 4.2 (i)], for which it is not clear that the original proof might be adapted, and we follow an alternative reasoning.

Proposition 3.1 *The right-hand-side of (8) remains equal if the maximum is restricted to*

$$\mathcal{M}_{\overline{p}}(\overline{x}) := \left\{D \in \mathcal{K}_{\overline{p}}(\overline{x}) \mid D \text{ is minimal for the inclusion order}\right\}.$$

Proof According to [6, Corollary 8], which is developed in the framework of canonical perturbations, the right hand side of (8) may be written as

$$\left(\|\overline{x}\| + 1\right) \mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right),$$

where $\mathcal{S}_{\overline{a}}(c, b) := \mathcal{S}(c, \overline{a}, b)$ for $(c, b) \in \mathbb{R}^n \times \mathbb{R}^T$.

In the referred framework of canonical perturbations, [5, Corollary 2] establishes that, adapted to our current notation and without assuming the uniqueness of nominal optimal solution,

$$\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = \max_{D \in \mathcal{M}_{\overline{p}}(\overline{x})} \mathrm{clm}\mathcal{L}_D\left(\overline{a}, \cdot, -\overline{a}_D, \cdot\right)\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right). \tag{9}$$

Finally, by applying [3, Theorem 3.1] (see also the proof of [3, Proposition 4.1], (9) may be rewritten as

$$\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) = \max_{D \in \mathcal{M}_{\overline{p}}(\overline{x})} \frac{1}{d_*\left(0_n, \mathrm{bd}\, C_D\right)}.$$

□

Corollary 3.1 *Under the current assumptions we have*

$$\operatorname{clm}\mathcal{S}\left(\overline{p},\overline{x}\right) \leq \max_{D\in\mathcal{M}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\|+1}{d_*\left(0_n, \operatorname{bd} C_D\right)}. \tag{10}$$

Remark 3.1 According to the previous paragraphs (see also [4, Remark 4.2]), inequality (10) *may* be read as

$$\operatorname{clm}\mathcal{S}\left(\overline{p},\overline{x}\right) \leq \left(\|\overline{x}\|+1\right) \operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right). \tag{11}$$

This inequality could constitute a refinement of the expected result derived from [19, Lemma 2], which would replace $(\|\overline{x}\|+1)$ in (11) by $\max\{(\|(\overline{x},u)\|+1) : u$ is a dual solution of $P\left(\overline{p}\right)\}$.

For simplicity in the notation let us denote

$$\Lambda_{\overline{p}}\left(\overline{x}\right) := \underset{D\in\mathcal{M}_{\overline{p}}(\overline{x})}{\arg\min}\; d_*\left(0_n, \operatorname{bd} C_D\right).$$

Observe that, under the current notation, (10) reads as

$$\operatorname{clm}\mathcal{S}(\left(\overline{c},\overline{a},\overline{b}\right),\overline{x}) \leq \frac{\|\overline{x}\|+1}{d_*\left(0_n, \operatorname{bd} C_D\right)} \text{ for any } D \in \Lambda_{\overline{p}}\left(\overline{x}\right).$$

The following example concerns the same nominal problem as [4, Example 4.1], which was used in that paper to show that inequality (8) may be strict. We will come back to this example in Section 6. At this moment we use it for illustrating sets $\mathcal{K}_{\overline{p}}\left(\overline{x}\right)$, $\mathcal{M}_{\overline{p}}\left(\overline{x}\right)$, and $\Lambda_{\overline{p}}\left(\overline{x}\right)$.

Example 3.1 Consider the nominal problem, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lll} P\left(\overline{c},\overline{a},\overline{b}\right): & \text{minimize} & 10x_1 \\ & \text{subject to} & -x_1+x_2 \leq -1 \;\; (t=1), \\ & & -2x_1-2x_2 \leq -6 \;\; (t=2), \\ & & -x_1 \leq -2 \;\; (t=3), \end{array}$$

whose unique optimal solution is $\overline{x}=(2,1)'$. Setting once more $\overline{p}=\left(\overline{c},\overline{a},\overline{b}\right)$, the reader can check the following:

$D \in \mathcal{K}_{\overline{p}}\left(\overline{x}\right)$	$\operatorname{clm}\mathcal{L}_D\left(\left(\overline{a},\overline{b},-\overline{a}_D,-\overline{b}_D\right),\overline{x}\right)$
$\{3\}$, $\{1,3\}$	$5+\sqrt{5}\approx 7.2361$
$\{1,2\}$	$\sqrt{10}\left(1+\sqrt{5}\right)/4 \approx 2.5583$
$\{2,3\}$	$\sqrt{13}\left(1+\sqrt{5}\right)/2 = 5.8339$

Accordingly, $\mathcal{M}_{\overline{p}}\left(\overline{x}\right)=\{\{3\},\{1,2\}\}$ and $\Lambda_{\overline{p}}\left(\overline{x}\right)=\{\{3\}\}$.

4 Lower Bound on the Calmness Modulus

Theorem 4.1 provides a lower bound on $\operatorname{clm}\mathcal{S}\left(\overline{p},\overline{x}\right)$ which turns out to be crucial for obtaining, in Corollary 4.1, more operative sufficient conditions under which the upper bound in Corollary 3.1 is attained. At this point we need some more notation. Although the statement of Theorem 4.1 is quite technical, its proof encloses important perturbation ideas, which are exploited in the rest of the paper.

For any given $x \in \mathbb{R}^n$ and any $D \in \mathcal{M}_{\overline{p}}(\overline{x})$ we define

$$U(x) := \left\{u \in \mathbb{R}^n \mid \|u\|_* = 1,\ u'x = \|x\|\right\},$$

$$\lambda_{t,x} := \frac{\overline{b}_t - \overline{a}_t' x}{\|x\| + 1},\ t \in T,$$

$$J_D(x) := \bigcup_{u \in U(x)} \text{cone}\left\{\overline{a}_t + \lambda_{t,x} u,\ t \in D\right\}.$$

Note that, as a straightforward consequence of the definition, $\lambda_{t,\overline{x}} = 0$ for all $t \in T_{\overline{a},\overline{b}}(\overline{x})$.

Theorem 4.1 *We have*

$$\text{clm}\mathcal{S}(\overline{p}, \overline{x}) \geq \max_{D \in \mathcal{M}_{\overline{p}}(\overline{x})} \limsup_{x \to \overline{x},\ x \neq \overline{x}} \frac{\|x - \overline{x}\|}{\max\{d_*(-\overline{c}, J_D(x)),\ f_D(x)/(\|x\| + 1)\}}.$$

Proof Fix arbitrarily any $D \in \mathcal{M}_{\overline{p}}(\overline{x})$ and consider sequence $\mathbb{R}^n \backslash \{\overline{x}\} \ni x^r \to \overline{x}$ such that

$$\gamma_D := \limsup_{x \to \overline{x},\ x \neq \overline{x}} \frac{\|x - \overline{x}\|}{\max\{d_*(-\overline{c}, J_D(x)),\ f_D(x)/(\|x\| + 1)\}}$$
$$= \lim_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\max\{d_*(-\overline{c}, J_D(x^r)),\ f_D(x^r)/(\|x^r\| + 1)\}}.$$

We are going to build a sequence $\{p^r\}$ converging to $\overline{p}$, with $x^r \in \mathcal{S}(p^r)$ for all r, and such that $\limsup_{r \to \infty} \|x^r - \overline{x}\| / \|p^r - \overline{p}\| \geq \gamma_D$. For each r let $u^r \in U(x^r)$ and $c^r \in -\text{cone}\left\{\overline{a}_t + \lambda_{t,x^r} u^r,\ t \in D\right\}$ (finitely generated and hence closed) be such that

$$\left\|\overline{c} - c^r\right\|_* = d_*\left(-\overline{c}, \text{cone}\left\{\overline{a}_t + \lambda_{t,x^r} u^r,\ t \in D\right\}\right) \tag{12}$$
$$\leq \frac{r+1}{r} d_*\left(-\overline{c}, J_D\left(x^r\right)\right),$$

and define $(a^r, b^r) \equiv \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}_{t \in T}$ as

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} := \begin{cases} \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} + \lambda_{t,x^r} \begin{pmatrix} u^r \\ -1 \end{pmatrix} & \text{if } t \in D \text{ or } \overline{a}_t' x^r > \overline{b}_t, \\ \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} & \text{if } \overline{a}_t' x^r \leq \overline{b}_t \text{ and } t \notin D. \end{cases}$$

Note that all $t \in T \backslash T_{\overline{a},\overline{b}}(\overline{x})$ belong to the latter case for r large enough. The reader can easily check from the definitons that $x^r \in \mathcal{L}_D\left(a^r, b^r, -a_D^r, -b_D^r\right)$ and

$$\left\|\left(a^r, b^r\right) - \left(\overline{a}, \overline{b}\right)\right\| = \frac{f_D(x^r)}{\|x^r\| + 1}. \tag{13}$$

Moreover, the fact that $c^r \in -\text{cone}\left\{\overline{a}_t + \lambda_{t,x^r} u^r,\ t \in D\right\}$ entails that x^r satisfies the KKT conditions for problem $P(c^r, a^r, b^r)$ with D as a KKT index set. Accordingly, $x^r \in \mathcal{S}(p^r)$, and (12) and (13) yield

$$\frac{\|x^r - \overline{x}\|}{\|p^r - \overline{p}\|} \geq \frac{\|x^r - \overline{x}\|}{\max\left\{\frac{r+1}{r} d_*(-\overline{c}, J_D(x^r)),\ f_D(x^r)/(\|x^r\| + 1)\right\}}$$
$$\geq \frac{r}{r+1} \frac{\|x^r - \overline{x}\|}{\max\{d_*(-\overline{c}, J_D(x^r)),\ f_D(x^r)/(\|x^r\| + 1)\}},$$

which implies $\text{clm}\mathcal{S}(\overline{p},\overline{x}) \geq \limsup_{r\to\infty} \|x^r - \overline{x}\| / \|p^r - \overline{p}\| \geq \gamma_D$. This finishes the proof, recalling that $D \in \mathcal{M}_{\overline{p}}(\overline{x})$ has been arbitrarily chosen. □

Corollary 4.1 *Assume that either* $-\overline{c} \in [0,1]\,\text{conv}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ *or* $-\overline{c} \in \text{int cone}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ *for some* $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$. *Then*

$$\text{clm}\mathcal{S}(\overline{p},\overline{x}) = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \text{bd}\, C_{\widehat{D}}\right)}.$$

Proof Consider first the case $-\overline{c} \in [0,1]\,\text{conv}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ for a certain $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$, and write

$$\overline{c} := -\sum_{t\in\widehat{D}} \mu_t \overline{a}_t$$

with $\mu_t \geq 0$ for all $t \in \widehat{D}$ and $\sum_{t\in\widehat{D}} \mu_t \leq 1$. Let us see that, for any $x \in \mathbb{R}^n$ one has $d_*\left(-\overline{c}, J_{\widehat{D}}(x)\right) \leq f_{\widehat{D}}(x) / (\|x\| + 1)$. To do this, take any $u \in U(x)$ and define

$$-c := \sum_{t\in\widehat{D}} \mu_t \left(\overline{a}_t + \lambda_{t,x} u\right),$$

which clearly belongs to $J_{\widehat{D}}(x)$ and verifies

$$\|-\overline{c} + c\|_* \leq \max_{t\in\widehat{D}} \left|\lambda_{t,x}\right| \leq f_{\widehat{D}}(x) / (\|x\| + 1).$$

Appealing now to Theorem 4.1 together with Remark 2.1 and Proposition 2.2, we conclude

$$\begin{aligned}
\text{clm}\mathcal{S}(\overline{p},\overline{x}) &\geq \limsup_{x\to\overline{x},\ x\neq\overline{x}} \frac{\|x - \overline{x}\|}{f_{\widehat{D}}(x) / (\|x\| + 1)} \\
&= \text{clm}\mathcal{L}_{\widehat{D}}\left(\left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right), \overline{x}\right) \\
&= \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \text{bd}\, C_{\widehat{D}}\right)} \geq \text{clm}\mathcal{S}(\overline{p},\overline{x}),
\end{aligned}$$

where we have taken $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$ into account. Recall that the last inequality of the previous chain is nothing else but (10).

Finally let us consider the case when $-\overline{c} \in \text{int cone}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ for a certain $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$. In this case we have $d_*\left(-\overline{c}, J_{\widehat{D}}(x)\right) = 0$ for x close enough to $\overline{x}$. To see this, just recall the definition of $J_{\widehat{D}}(x)$ and observe that, for each $t \in \widehat{D}$, one has $\lambda_{t,x} \to 0$ as $x \to \overline{x}$ (see, for instance, [10, Exercise 6.12]). □

Remark 4.1 Taking our uniqueness assumption $\mathcal{S}(\overline{p}) = \{\overline{x}\}$ into account, if the Linear Independence Constraint Qualification (LICQ) is satisfied at $\overline{x}$ for our nominal system, i.e., $\left\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\right\}$ is linearly independent, then we have

$$\Lambda_{\overline{p}}(\overline{x}) = \mathcal{K}_{\overline{p}}(\overline{x}) = \left\{T_{\overline{a},\overline{b}}(\overline{x})\right\},$$

and the second case of the previous corollary occurs.

5 Critical Objective Size

In this section we are going to show that, if $\|\overline{c}\|_*$ is small enough, then the upper bound (10) is attained. Define

$$\overline{p}_k := \left(k\overline{c}, \overline{a}, \overline{b}\right) \text{ for } k > 0.$$

Clearly $\mathcal{S}\left(\overline{p}_k\right) = \{\overline{x}\}$ and $\mathcal{M}_{\overline{p}_k}(\overline{x}) = \mathcal{M}_{\overline{p}}(\overline{x})$ (and hence $\Lambda_{\overline{p}_k}(\overline{x}) = \Lambda_{\overline{p}}(\overline{x})$) for all $k > 0$. Define as well

$$\mathcal{A}_{\overline{p}}(\overline{x}) := \left\{k > 0 \mid \operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right) = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{bd} C_D\right)} \text{ for some } D \in \Lambda_{\overline{p}}(\overline{x})\right\}.$$

Obviously, 'for some' could be replaced with 'for all' in the definition of $\mathcal{A}_{\overline{p}}(\overline{x})$. The next result shows the monotonic behavior of $\operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right)$ with respect to k.

Proposition 5.1 $\operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right) \geq \operatorname{clm}\mathcal{S}\left(\overline{p}_{k_0}, \overline{x}\right)$ *whenever* $0 < k < k_0$. *Consequently, if* $k_0 \in \mathcal{A}_{\overline{p}}(\overline{x})$, *then* $k \in \mathcal{A}_{\overline{p}}(\overline{x})$ *for all* $k \in \left]0, k_0\right[$.

Proof Let us write $\operatorname{clm}\mathcal{S}\left(\overline{p}_{k_0}, \overline{x}\right) = \lim_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|p^r - \overline{p}_{k_0}\right\|}$ for certain sequences of parameters $p^r = (c^r, a^r, b^r)$ and points $x^r \in \mathcal{S}(p^r)$ such that $(p^r, x^r) \to \left(\overline{p}_{k_0}, \overline{x}\right)$ with $p^r \neq \overline{p}_{k_0}$. Take any $k \in \left]0, k_0\right[$. Then, since obviously $x^r \in \mathcal{S}\left(kk_0^{-1}c^r, a^r, b^r\right)$, $\left(kk_0^{-1}c^r, a^r, b^r\right) \to \overline{p}_k$ as $r \to \infty$, and, directly from the definitions of the norms involved, $\left\|\left(kk_0^{-1}c^r, a^r, b^r\right) - \overline{p}_k\right\| \leq \left\|p^r - \overline{p}_{k_0}\right\|$, we conclude

$$\operatorname{clm}\mathcal{S}\left(\overline{p}_{k_0}, \overline{x}\right) \leq \limsup_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|\left(kk_0^{-1}c^r, a^r, b^r\right) - \overline{p}_k\right\|} \leq \operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right).$$

□

The next proposition ensures the nonemptiness of $\mathcal{A}_{\overline{p}}(\overline{x})$.

Proposition 5.2 *The following conditions hold:*

(i) *Let* $-\overline{c} = \sum_{t\in\widehat{D}} \lambda_t \overline{a}_t$ *for some* $(\lambda_t)_{t\in\widehat{D}} \in \mathbb{R}_+^{\widehat{D}}$ *and some* $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$. *Then*

$$\left(\sum_{t\in\widehat{D}} \lambda_t\right)^{-1} \in \mathcal{A}_{\overline{p}}(\overline{x}).$$

(ii) *If* $-\overline{c} \in \operatorname{int}\operatorname{cone}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ *for some* $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$, *then* $\mathcal{A}_{\overline{p}}(\overline{x}) = \left]0, +\infty\right[$.

Proof Both statements are straightforward consequences of Corollary 4.1. □

Definition 5.1 The critical objective size of problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$, at $\overline{x}$, is defined as

$$\tau_{\overline{p}}(\overline{x}) := \|\overline{c}\|_* \sup \mathcal{A}_{\overline{p}}(\overline{x}),$$

understood as $+\infty$ if $\mathcal{A}_{\overline{p}}(\overline{x}) = \left]0, +\infty\right[$.

Remark 5.1 As a direct consequence of Proposition 5.2 (i), if $-\overline{c} = \sum_{t\in\widehat{D}} \lambda_t \overline{a}_t$ for some $(\lambda_t)_{t\in\widehat{D}} \in \mathbb{R}_+^{\widehat{D}}$ and some $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$, then we have

$$\tau_{\overline{p}}(\overline{x}) \geq \left(\sum_{t\in\widehat{D}} \lambda_t\right)^{-1} \|\overline{c}\|_* .$$

On the other hand, if $-\overline{c} \in \operatorname{int}\operatorname{cone}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ for some $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$, then $\tau_{\overline{p}}(\overline{x}) = +\infty$. This is the case when LICQ holds at $\overline{x}$ for our nominal system (see Remark 4.1).

Proposition 5.3 *The supremum* $\sup \mathcal{A}_{\overline{p}}(\overline{x})$, *when finite, is attained.*

Proof We can follow a sort of diagonal process. Let $k_0 = \sup \mathcal{A}_{\overline{p}}(\overline{x}) \in \mathbb{R}$ and take any sequence $\{k_r\}_{r\in\mathbb{N}} \subset \mathcal{A}_{\overline{p}}(\overline{x})$ converging to k_0. Pick any $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$. For each r, write

$$\operatorname{clm}\mathcal{S}\left(\overline{p}_{k_r}, \overline{x}\right) = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{bd} C_{\widehat{D}}\right)} = \lim_{s\to\infty} \frac{\|x^{r,s} - \overline{x}\|}{\left\|p^{r,s} - \overline{p}_{k_r}\right\|}$$

(see the comment right after the definition of $\mathcal{A}_{\overline{p}}(\overline{x})$) for certain sequences of parameters $p^{r,s}$ and points $x^{r,s} \in \mathcal{S}(p^{r,s})$ such that $(p^{r,s}, x^{r,s}) \to \left(\overline{p}_{k_r}, \overline{x}\right)$ as $s \to \infty$, with $p^{r,s} \neq \overline{p}_{k_r}$ for all s. Take, for each r, a certain $s_r > r$ such that $\|x^{r,s_r} - \overline{x}\| < \frac{1}{r}$, $\left\|p^{r,s_r} - \overline{p}_{k_r}\right\| < \frac{1}{r}$, and

$$\left| \frac{\|x^{r,s_r} - \overline{x}\|}{\left\|p^{r,s_r} - \overline{p}_{k_r}\right\|} - \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{bd} C_{\widehat{D}}\right)} \right| < \frac{1}{r}. \tag{14}$$

Now write for simplicity x^r instead of x^{r,s_r} and $p^r = (c^r, a^r, b^r)$ instead of p^{r,s_r}. Define, for each r, $\widetilde{p}^r := \left(\frac{k_0}{k_r} c^r, a^r, b^r\right)$. Then we can write

$$\begin{aligned} \left\|\widetilde{p}^r - \overline{p}_{k_0}\right\| &= \left\| \left(\frac{k_0}{k_r} c^r, a^r, b^r\right) - \left(\frac{k_0}{k_r} k_r \overline{c}, \overline{a}, \overline{b}\right) \right\| \\ &= \max\left\{ \frac{k_0}{k_r} \left\|c^r - k_r \overline{c}\right\|_*, \left\|\left(a^r, b^r\right) - \left(\overline{a}, \overline{b}\right)\right\| \right\}, \end{aligned}$$

from which we easily get

$$\left\|p^r - \overline{p}_{k_r}\right\| \leq \left\|\widetilde{p}^r - \overline{p}_{k_0}\right\| \leq \frac{k_0}{k_r} \left\|p^r - \overline{p}_{k_r}\right\| .$$

This fact together with (14) entails

$$\operatorname{clm}\mathcal{S}\left(\overline{p}_{k_0}, \overline{x}\right) \geq \lim_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|\widetilde{p}^r - \overline{p}_{k_0}\right\|} = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{bd} C_{\widehat{D}}\right)},$$

which completes the proof of the proposition. □

Remark 5.2 We have $\tau_{k\overline{p}}(\overline{x}) = \tau_{\overline{p}}(\overline{x})$ for all $k > 0$; i.e., the critical objective size does not depend on $\|\overline{c}\|_*$. As an immediate consequence of the definition we conclude that the upper bound (10) on $\operatorname{clm}\mathcal{S}(\overline{p}, \overline{x})$ is attained if and only if $\|\overline{c}\|_* \leq \tau_{\overline{p}}(\overline{x})$. In particular, as an immediate consequence of Corollary 4.1, if $-\overline{c} \in \operatorname{conv}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ for some $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$, then $1 \in \mathcal{A}_{\overline{p}}(\overline{x})$ and, hence, $\|\overline{c}\|_* \leq \tau_{\overline{p}}(\overline{x})$ occurs.

Remark 5.3 Corollary 4.1 also ensures that $\tau_{\overline{p}}(\overline{x}) = +\infty$ in the case when $-\overline{c} \in \operatorname{int}\operatorname{cone}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}$ for some $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$.

The following theorem provides, in terms of the critical objective size, a lower bound on $\text{clm}\mathcal{S}(\overline{p}, \overline{x})$ which only depends on the nominal data $\overline{p}$ and $\overline{x}$.

Theorem 5.1 *The following condition holds:*

$$\text{clm}\mathcal{S}(\overline{p}, \overline{x}) \geq \max_{D \in \mathcal{M}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\| + 1}{d_* (0_n, \text{bd}\, C_D) \max\left\{1, \|\overline{c}\|_* / \tau_{\overline{p}}(\overline{x})\right\}}.$$

Proof According to Remark 5.2, we just have to prove the case when $\|\overline{c}\|_* > \tau_{\overline{p}}(\overline{x})$. In this case, for $\overline{k} := \tau_{\overline{p}}(\overline{x}) / \|\overline{c}\|_* < 1$ we have $1 \in \mathcal{A}_{\overline{p}_{\overline{k}}}(\overline{x})$ (see Proposition 5.3). This means $\text{clm}\mathcal{S}\left(\overline{p}_{\overline{k}}, \overline{x}\right) = \max_{D \in \mathcal{M}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\|+1}{d_*(0_n, \text{bd}\, C_D)}$. Now write

$$\text{clm}\mathcal{S}\left(\overline{p}_{\overline{k}}, \overline{x}\right) = \lim_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\left\|p^r - \overline{p}_{\overline{k}}\right\|}$$

for some sequences $p^r := (c^r, a^r, b^r) \to \overline{p}_{\overline{k}} = \left(\overline{k}\overline{c}, \overline{a}, \overline{b}\right)$ and $\mathcal{S}(p^r) \ni x^r \to \overline{x}$. Then we have

$$\begin{aligned}
\text{clm}\mathcal{S}(\overline{p}, \overline{x}) &\geq \limsup_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\left\|\left(\overline{k}^{-1} c^r, a^r, b^r\right) - \overline{p}\right\|} \\
&= \limsup_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\max\left\{\overline{k}^{-1}\left\|c^r - \overline{k}\overline{c}\right\|_*, \left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\|\right\}} \\
&\geq \limsup_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\overline{k}^{-1} \max\left\{\left\|c^r - \overline{k}\overline{c}\right\|_*, \left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\|\right\}} \\
&= \overline{k}\,\text{clm}\mathcal{S}\left(\overline{p}_{\overline{k}}, \overline{x}\right) = \max_{D \in \mathcal{M}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\| + 1}{d_*(0_n, \text{bd}\, C_D)\left(\|\overline{c}\|_* / \tau_{\overline{p}}(\overline{x})\right)}.
\end{aligned}$$

□

6 Perturbation Strategies for Improved Estimates

The lower bound on $\tau_{\overline{p}}(\overline{x})$ given in Remark 5.1 has the virtue of relying exclusively on the nominal data $\overline{p}$ and $\overline{x}$ (we could indeed consider the best choice of $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$ for this). Nevertheless, Theorem 4.1 provides the following strategy to improve such a lower bound:

1. Choose any $\widehat{D} \in \Lambda_{\overline{p}}(\overline{x})$ and write

$$\text{clm}\mathcal{L}_{\widehat{D}}\left(\left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right), \overline{x}\right) = \lim_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\|}$$

for suitable sequences $\{(a^r, b^r)\} \subset (\mathbb{R}^n)^T \times \mathbb{R}^T$ and $\{x^r\} \subset \mathbb{R}^n$ such that

$$x^r \in \mathcal{L}_{\widehat{D}}\left(a^r, b^r, -a^r_{\widehat{D}}, -b^r_{\widehat{D}}\right) \text{ and } \left\|\left(a^r, b^r\right) - \left(\overline{a}, \overline{b}\right)\right\| = \frac{1}{r} \text{ for all } r \in \mathbb{N}.$$

2. Find $k > 0$ and $\widetilde{c}^r \in \text{cone}\left\{-a^r_t,\ t \in \widehat{D}\right\}$ such that

$$\left\|k\overline{c} - \widetilde{c}^r\right\|_* = d_*\left(k\overline{c}, \text{cone}\left\{-a^r_t,\ t \in \widehat{D}\right\}\right) \approx \frac{1}{r},$$

i.e., $\lim_{r\to\infty} r\,\|k\overline{c}-\widetilde{c}^r\|_* = 1$ (of course, $\frac{1}{r}$ can be replaced from the beginning of the proof with any $\varepsilon_r \downarrow 0$).

3. Then, such a k belongs to $\mathcal{A}_{\overline{p}}(\overline{x})$.

Now we come back to Example 3.1, where we provide lower and upper estimations of $\tau_{\overline{p}}(\overline{x})$, as well as sharper lower and upper bounds on $\operatorname{clm}\mathcal{S}(\overline{p},\overline{x})$ than those given in Theorem 5.1 and Corollary 3.1, respectively. Some technical details given below show a strong parallelism with [4, Section 6], and the reader is addressed there for a complete discussion.

Example 6.1 Consider the nominal problem $P\left(\overline{c},\overline{a},\overline{b}\right)$ given in Example 3.1. We point out the following facts:

(i) By applying the previous strategy with $\widehat{D}=\{3\}$ and the same $\left\{\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}\right\}_{t\in\{1,2,3\}}$ defined in [4, Example 4.2], we obtain

$$d_*\left(k\overline{c}, \operatorname{cone}\left\{-a_t^r,\ t\in\widehat{D}\right\}\right) \approx \frac{10k}{r\sqrt{5}}.$$

Accordingly, $k := 1/\left(2\sqrt{5}\right) \in \mathcal{A}_{\overline{p}}(\overline{x})$, which entails $\tau_{\overline{p}}(\overline{x}) \geq \sqrt{5}$.

(ii) By replacing '10' with '$10k$' in the system given in [4, Equation (27)] we obtain, with the corresponding counterpart of the point $\widetilde{x}^r$ defined just before [4, Equation (28)], for any $k>0$,

$$\operatorname{clm}\mathcal{S}\left(\overline{p}_k,\overline{x}\right) \leq \lim_{r\to\infty}\frac{\|\widetilde{x}^r-\overline{x}\|}{\alpha_r} = \varphi(k) := \sqrt{8\sqrt{5}+30+\tfrac{7+\sqrt{5}}{5k}+\tfrac{1}{50k^2}}.$$

Then we see that φ is a strictly decreasing function with $\varphi^{-1}\left(5+\sqrt{5}\right) = \left(2+\sqrt{5}\right)/10$, which entails $\tau_{\overline{p}}(\overline{x}) \leq 2+\sqrt{5}$.

(iii) Also observe that $\lim_{k\to+\infty}\operatorname{clm}\mathcal{S}\left(\overline{p}_k,\overline{x}\right) \leq \lim_{k\to\infty}\varphi(k) = \sqrt{8\sqrt{5}+30}$ (quantity which appears at the end of [4, Section 6]). Note that the first limit exists since the function $k\mapsto \operatorname{clm}S\left(\overline{p}_k,\overline{x}\right)$ is decreasing according to Proposition 5.1. We will come to this point later.

From the beginning of this section we are considering the strategy of perturbing $\overline{x}$ and $\left(\overline{a},\overline{b}\right)$ in order to obtain x^r and (a^r,b^r) such that $\operatorname{clm}\mathcal{L}_{\widehat{D}}\left(\left(\overline{a},\overline{b},-\overline{a}_{\widehat{D}},-\overline{b}_{\widehat{D}}\right),\overline{x}\right)$, for a given $\widehat{D}\in\Lambda_{\overline{p}}(\overline{x})$, may be written as $\lim_{r\to\infty}\|x^r-\overline{x}\|/\left\|(a^r,b^r)-\left(\overline{a},\overline{b}\right)\right\|$, and then perturbing $\overline{c}$ in such a way that the perturbed x^r becomes optimal for the perturbed parameter (c^r,a^r,b^r). The drawback of this strategy is that the perturbation size $\|c^r-\overline{c}\|_*$ might be essentially larger than $\left\|(a^r,b^r)-\left(\overline{a},\overline{b}\right)\right\|$, which would spoil the limit of the ratio when replacing $\left\|(a^r,b^r)-\left(\overline{a},\overline{b}\right)\right\|$ with $\left\|(c^r,a^r,b^r)-\left(\overline{c},\overline{a},\overline{b}\right)\right\|$. An alternative strategy to prevent this situation consists of making a smaller perturbation on those $\overline{a}_t$ with $t\in\widehat{D}$ in order to need a smaller perturbation of $\overline{c}$ to get the KKT conditions at the perturbed x^r. Roughly speaking, instead of just moving $k\overline{c}$ towards cone $\left\{-a_t^r,\ t\in\widehat{D}\right\}$, we could move $k\overline{c}$ to the new perturbed cone and, at the same time, move the cone towards $k\overline{c}$. We try to illustrate these ideas in the next example, which leans again on [4, Example 4.1 and Section 6].

Example 6.2 Consider again the nominal problem Example 6.1, as well as $\widehat{D} = \{3\}$. Recall that we are dealing with the Euclidean norm. Consider the same a_t^r for $t \in \{1, 2\}$ and the same b_t^r for $t \in \{1, 2, 3\}$ as in Example 6.1, and set

$$a_3^r := \overline{a}_3 + \frac{1}{r}v \textit{ with } \|v\| = 1,\ v = (v_1, v_2),\ v_1 > 0,\ v_2 > 0,$$
$$c^r \in \operatorname{cone}\left\{-a_3^r\right\} \textit{ such that } \left\|k\overline{c} - c^r\right\| = d\left(k\overline{c}, \operatorname{cone}\left\{-a_3^r,\ t \in \widehat{D}\right\}\right) \approx \frac{1}{r}.$$

With our current data, this entails $v = \left(\sqrt{1-(10k)^{-2}}, (10k)^{-1}\right)$ with $k > 1/10$. Then we define x^r as the solution of the system $\left\{\left(a_t^r\right)' x = b_t^r,\ t = 1, 3\right\}$. It can be checked that, for any given $k > 1/10$, $x^r \in \mathcal{S}(c^r, a^r, b^r)$ for r large enough. Thus,

$$\operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right) \geq \psi(k) := \lim_{r\to\infty} r\left\|x^r - \overline{x}\right\|,$$

and after some calculations we obtain

$$\psi(k) = \sqrt{\frac{10\left(\sqrt{5}+3\right)+\frac{2}{k}}{25}\sqrt{100-\frac{1}{k^2}} + 4\sqrt{5} + 18 + \frac{\sqrt{5}+3}{5k} - \frac{3}{50k^2}},$$

which is strictly increasing in $\left]1/10, 1/\left(2\sqrt{5}\right)\right]$ and strictly decreasing in $\left[1/\left(2\sqrt{5}\right), +\infty\right[$. Moreover,

$$\psi\left(1/\left(2\sqrt{5}\right)\right) = 5 + \sqrt{5}, \tag{15}$$

$$\lim_{k\to+\infty} \psi(k) = \sqrt{8\sqrt{5}+30} \approx 6.9202. \tag{16}$$

From (15) we conclude $1/\left(2\sqrt{5}\right) \in \mathcal{A}_{\overline{p}}(\overline{x})$, which we already knew and entails $\tau_{\overline{p}}(\overline{x}) \geq \sqrt{5}$. Indeed, for our nominal problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$, and recalling function φ in Example 6.1 (ii), we deduce that

$$7.0404 \approx \psi(1) \leq \operatorname{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq (1/10)\sqrt{820\sqrt{5}+3142} \leq \varphi(1) \approx 7.0538$$

(the latter inequality was already known from [4, Example 4.1]). Finally, (16) ensures that $\sqrt{8\sqrt{5}+30} \approx 6.9202$ is a lower bound on $\operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right)$ for all $k > 0$. This together with Example 6.1 (iii) ensures that

$$\lim_{k\to+\infty} \operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right) = \sqrt{8\sqrt{5}+30}.$$

For $k > 1/\left(2\sqrt{5}\right)$ we only can ensure that

$$\psi(k) \leq \operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right) \leq \min\left\{\varphi(k), 5+\sqrt{5}\right\},$$

but computing the exact value of $\operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right)$ remains as an open problem. Recall that $\operatorname{clm}\mathcal{S}\left(\overline{p}_k, \overline{x}\right) = 5 + \sqrt{5}$ whenever $0 < k \leq 1/\left(2\sqrt{5}\right)$.

The conclusions about the previous example are summarized in the following figure (Fig. 1):

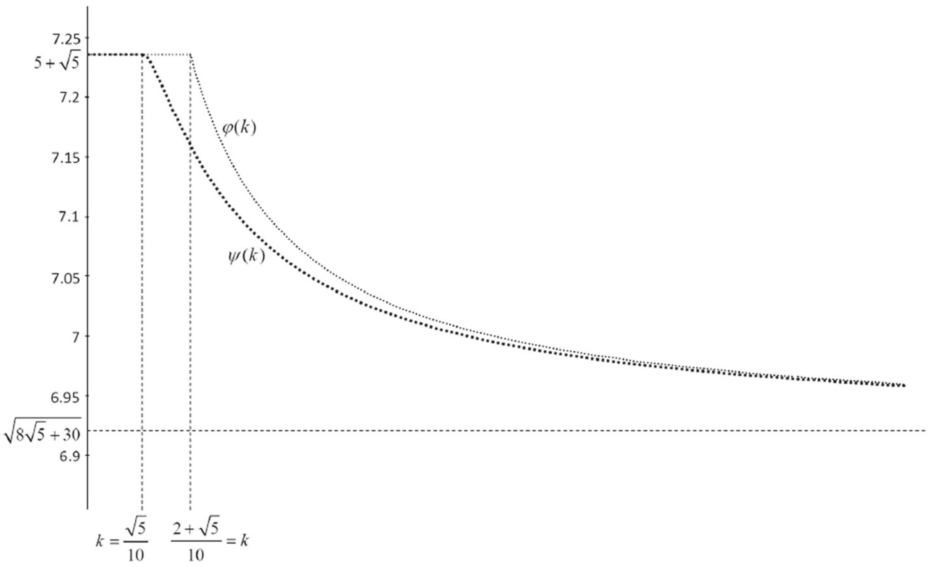

Fig. 1 Bounds on the calmness modulus in terms of the objective size

7 Conclusions

In this section, we summarize the main contributions of the paper. Our starting point is the upper bound (8) on $\text{clm}\mathcal{S}(\overline{p}, \overline{x})$, with $\overline{p} = (\overline{c}, \overline{a}, \overline{b})$ and $\mathcal{S}(\overline{p}) = \{\overline{x}\}$, given in [4, Theorem 4.2(i)]. Proposition 3.1 shows that the right-hand-side of (8) remains equal if the maximum is restricted to minimal KKT index sets, leading to (10). In Theorem 4.1 we provide a technical lower bound which leads to sufficient conditions for equality in (10), see Corollary 4.1. Roughly speaking, the upper bound is attained if and only if the size of the objective function coefficient vector $\overline{c}$ is small enough. In order to formalize this assertion we introduce in Definition 5.1 the concept of critical objective size, $\tau_{\overline{p}}(\overline{x})$, and prove that (10) becomes an equality if and only if $\|\overline{c}\|_* \leq \tau_{\overline{p}}(\overline{x})$, see Remark 5.2. Moreover, in terms of $\tau_{\overline{p}}(\overline{x})$ we are able to provide a more operative lower bound on $\text{clm}\mathcal{S}(\overline{p}, \overline{x})$, see Theorem 5.1. Finally, in Section 6 we illustrate by means of examples some perturbation strategies which may lead to tighter bounds on $\text{clm}\mathcal{S}(\overline{p}, \overline{x})$.

Obtaining an operative expression for $\tau_{\overline{p}}(\overline{x})$ in terms of the nominal data remains as an open problem. In Examples 6.1 and 6.2 (both tackling the same optimization problem) we are able to provide lower and upper estimates on such a quantity $\tau_{\overline{p}}(\overline{x})$, as well as lower and upper estimates on $\text{clm}\mathcal{S}((k\overline{c}, \overline{a}, \overline{b}), \overline{x})$ in terms of k, both estimates having the same asymptotic value. Obtaining an operative exact expression for $\text{clm}\mathcal{S}(\overline{p}, \overline{x})$ when $\|\overline{c}\|_* > \tau_{\overline{p}}(\overline{x})$ also remains as an open problem.

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