Journal of Mathematical Analysis and Applications **258,** 110–130 (2001)
doi:10.1006/jmaa.2000.7363, available online at http://www.idealibrary.com on IDEAL®

A Subdifferential Condition for Calmness of Multifunctions

René Henrion[1]

Weierstrass Institute Berlin, 10117 Berlin, Germany

and

Jirí Outrata[2]

Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 18208 Prague, Czech Republic
E-mail: outrata@utia.cas.cz

Submitted by Boris S. Mordukhovich

Received March 27, 2000

A condition ensuring calmness of a class of multifunctions between finite-dimensional spaces is derived in terms of subdifferential concepts developed by Mordukhovich. The considered class comprises general constraint set mappings as they occur in optimization or mappings associated with a certain type of variational system. The condition ensuring calmness is obtained as an appropriate reduction of Mordukhovich's well-known characterization of the stronger Aubin property. (Roughly spoken, one may pass to the boundaries of normal cones or subdifferentials when aiming at calmness.) It allows one to derive dual constraint qualifications in nonlinear optimization that are weaker than conventional ones (e.g., Mangasarian–Fromovitz) but still sufficient for the existence of Lagrange multipliers. © 2001 Academic Press

Key Words: calmness; constraint qualification; multifunctions; coderivative; constraint sets; nonlinear complementarity problems.

1. INTRODUCTION

Frequently, the stability analysis of multifunctions $M : Y \rightrightarrows X$ between metric spaces X, Y, relies on the Aubin property, which is said to hold at

[1] Author to whom correspondence should be sent.
[2] The research of this author was supported by Grant No. 201/00/0080 of the Grant Agency of the Czech Republic.

0022-247X/01 $35.00

some $(\bar{y}, \bar{x}) \in \text{Gph}\, M$ (= graph of M), if there exist neighborhoods $\mathscr{V}$, $\mathscr{U}$ of $\bar{y}$, $\bar{x}$, as well as some $L > 0$ such that

$$d(x, M(y_2)) \leq Ld(y_1, y_2) \quad \text{for all } y_1, y_2 \in \mathscr{V}, \quad \text{for all } x \in M(y_1) \cap \mathscr{U}.$$

This property is well known to be equivalent with the metric regularity of the inverse multifunction M^{-1} (cf., e.g., [9], Theorem 9.43). In case of finite-dimensional spaces X, Y, it is possible to characterize equivalently the Aubin property of closed multifunctions by the algebraic criterion (see [6])

$$D^*M(\bar{y}, \bar{x})(0) = \{0\}, \tag{1}$$

where D^*M refers to Mordukhovich's coderivative. A weaker concept of Lipschitz-like behavior of multifunctions is calmness, which is satisfied at some $(\bar{y}, \bar{x}) \in \text{Gph}\, M$, if there exist neighborhoods $\mathscr{V}$, $\mathscr{U}$ of $\bar{y}$, $\bar{x}$, as well as some $L > 0$ such that

$$d(x, M(\bar{y})) \leq Ld(y, \bar{y}) \quad \text{for all } y \in \mathscr{V}, \quad \text{for all } x \in M(y) \cap \mathscr{U}.$$

It is easy to see that this property is equivalent to a "nonparametric" form of the metric regularity of M^{-1} at $(\bar{x}, \bar{y})$, introduced in the special case of

$$M(y) = \{x \in \Omega \,|\, f(x) = y\}$$

(with a closed set Ω and a single-valued map f) in [3]. This weakened form of metric regularity implies that $d(M^{-1}(x), \bar{y})$ is a Lipschitzian error bound for the constraint

$$x \in M(\bar{y}),$$

provided that x is close to $\bar{x}$ and $d(M^{-1}(x), \bar{y})$ is sufficiently small.

The concept of calmness, applied to value functions of optimization problems, goes back to Clarke [1] and Rockafellar, who pointed out its relevance as a constraint qualification for obtaining nondegenerate Lagrange multipliers in optimization problems. To illustrate the analogous role that calmness of multifunctions plays in the same context, assume that M is a closed multifunction and that $\bar{x}$ is a local minimizer of some locally Lipschitzian function φ on $M(\bar{y})$. Then there is some $K > 0$ such that $\varphi(\bar{x}) \leq \varphi(x) + Kd(x, M(\bar{y}))$ for all x in a neighborhood of $\bar{x}$ (see Proposition 2.4.3 in [2]). Now if M is calm at $(\bar{y}, \bar{x})$, then the last inequality may be extended to $\varphi(\bar{x}) \leq \varphi(x) + KLd(y, \bar{y})$, which holds true for all $x \in M(y)$ with x close to $\bar{x}$ and y close to $\bar{y}$. This, however, is exactly the calmness condition shown in [2, Proposition 6.4.4] to yield a nonsmooth (nondegenerate) multiplier rule for finite-dimensional optimization problems with Lipschitzian data.

For the derivation of multiplier rules, it is usual to indicate appropriate constraint qualifications that have a chance to be verified for the given data. Frequently, such constraint qualifications are associated with the Aubin property rather than calmness of the underlying constraint set mapping. But this may result in too-strong conditions, as is most easily seen from the convex example, in which $\varphi(x) = x$ is minimized subject to $g(x) = |x| \leq 0$. Here the Aubin property of the constraint set mapping (which is equivalent to Slater's condition) fails to hold for the minimizer because $0 \in \partial g(0) = [-1, 1]$. On the other hand, calmness is fulfilled, and consequently, one has the multiplier rule $0 \in \partial\varphi(0) + \lambda\partial g(0) = [1 - \lambda, 1 + \lambda]$ for some $\lambda \geq 0$. An appropriate constraint qualification in this example would be the condition $0 \notin \operatorname{bd} \partial g(0)$, where bd refers to the topological boundary. This can be considered to be a weak Slater's condition, which is actually satisfied in the foregoing example.

To put this idea into a more general context, we consider the following class of finite-dimensional multifunctions:

$$M(y) := \{x \in \Omega | g(x) + y \in \Lambda\},$$

where $\Omega \subseteq \mathbb{R}^p, \Lambda \subseteq \mathbb{R}^m$ are closed subsets and $g : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. This class covers constraint sets of nonsmooth, finite-dimensional optimization but also some generalized equations—in particular, nonlinear complementarity problems. Applying the criterion (1) for the Aubin property to this structure gives at some $(0, \bar{x}) \in \operatorname{Gph} M$

$$\bigcup_{y^* \in N_\Lambda(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(y^*) \cap (-N_\Omega(\bar{x})) = \emptyset, \tag{2}$$

where N refers to Mordukhovich's normal cone. Now the main result of this paper states that under mild assumptions on Ω and g, the weaker calmness property can be guaranteed under the weaker condition

$$\bigcup_{y^* \in N_\Lambda(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(y^*) \cap (-\operatorname{bd} N_\Omega(\bar{x})) = \emptyset.$$

Indeed, this criterion applies without any further assumptions on g given that the abstract constraint set Ω (which typically has a simple structure) is convex or defined as an intersection or union of a finite number of smooth inequalities under the usual regularity condition. The result is no longer true for arbitrary closed sets Ω, but at least for those that are Clarke regular it can be saved under the additional assumption that either g or Ω is semismooth. Moreover, if g is Clarke regular in the special case $\Lambda = \mathbb{R}^m_-$ (modeling a finite number of inequalities), one can even sharpen the foregoing condition by passing to the boundary on the left side as well.

In this way, the simple convex example mentioned earlier will be covered, and more generally, new constraint qualifications ensuring Lagrange multipliers can be derived. Finally, the obtained condition is applied to the case of nonlinear complementarity problems.

2. BASIC CONCEPTS AND NOTATION

The following notation is used throughout this paper: $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^n$, $\mathbb{B}_2$ is the respective unit ball, $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$, $\|\cdot\|^*$ is the corresponding dual norm, and $\mathbb{B}^*$ is the unit ball associated with $\|\cdot\|^*$. For a set Ω, δ_Ω and d_Ω^e denote the indicator and the Euclidean distance function, respectively. Finally, $\sharp$ refers to the cardinality of sets.

Next we recall some basic concepts from nonsmooth analysis needed in this paper. For a closed subset $A \subseteq \mathbb{R}^k$, the contingent and Clarke's tangent cone, respectively, to A at some point $\bar{x} \in A$ are defined by

$$K_A(\bar{x}) = \left\{d \in \mathbb{R}^k | \exists t_n \downarrow 0,\, d_n \to d : \bar{x} + t_n d_n \in A\right\}$$

$$T_A^c(\bar{x}) = \left\{d \in \mathbb{R}^k | \forall t_n \downarrow 0,\, x_n \to \bar{x}(x_n \in A) \exists d_n \to d : x_n + t_n d_n \in A\right\}.$$

The respective normal cones are obtained as

$$\hat{N}_A(\bar{x})\left(N_A^c(\bar{x})\right) = \left\{d^* \in \mathbb{R}^k | \langle d^*, d\rangle \le 0\ \forall d \in K_A(\bar{x})\left(T_A^c(\bar{x})\right)\right\}.$$

In contrast, the Mordukhovich normal cone is defined as a generally nonconvex object via

$$N_A(\bar{x}) = \left\{d^* \in \mathbb{R}^k | \exists d_n^* \to d^*,\, x_n \to \bar{x}(x_n \in A) : d_n^* \in \hat{N}_A(x_n)\right\}.$$

A is called (Clarke-) regular at $\bar{x}$ if $K_A(\bar{x}) = T_A^c(\bar{x})$ or, equivalently, $N_A(\bar{x}) = N_A^c(\bar{x}) = \hat{N}_A(\bar{x})$. We let $\operatorname{epi} \varphi = \{(x, \alpha) \in \mathbb{R}^{k+1} | \varphi(x) \le \alpha\}$ denote the epigraph of a lower semicontinuous function $\varphi : \mathbb{R}^k \to \overline{\mathbb{R}}$. Now the normal cones induce subdifferentials of φ via

$$\partial\varphi(\bar{x})(\partial^c\varphi(\bar{x})) = \left\{x^* \in \mathbb{R}^k | (x^*, -1) \in N_{\operatorname{epi} \varphi}(\bar{x})\left(N^c_{\operatorname{epi} \varphi}(\bar{x})\right)\right\},$$

where ∂ and ∂^c refer to Mordukhovich and Clarke subdifferentials, respectively. A more general construction is the Mordukhovich coderivative $D^*Z(\bar{x}, \bar{y}) : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$ of some multifunction $Z : \mathbb{R}^k \to \mathbb{R}^l$ at some point $(\bar{x}, \bar{y}) \in \operatorname{cl} \operatorname{Gph} Z$,

$$D^*Z(\bar{x}, \bar{y})(y^*) = \left\{x^* \in \mathbb{R}^k | (x^*, -y^*) \in N_{\operatorname{Gph} Z}(\bar{x}, \bar{y})\right\},$$

where the argument $\bar{y}$ is omitted if Z is single-valued. For single-valued, locally Lipschitzian mappings $g = (g_1, \ldots, g_l) : \mathbb{R}^k \to \mathbb{R}^l$, the basic relation between a coderivative and subdifferential of its components is $D^*g(\bar{x})(y^*) = \partial(\sum_{i=1}^{l} y_i^* g_i)(\bar{x})$. For a detailed treatment of the objects mentioned here, see [9], [2], and [7].

For technical reasons, we use the concept of semismooth functions introduced by Mifflin in [4].

DEFINITION 2.1. A function $\psi : \mathbb{R}^k \to \mathbb{R}$ is called semismooth at $\bar{x} \in \mathbb{R}^k$ if it is locally Lipschitz at $\bar{x}$ and the following property holds true: For each $d \in \mathbb{R}^k$ and for any sequences $t_n \downarrow 0$, $d_n \to d$, $x_n^* \in \partial^c \psi(\bar{x} + t_n d_n)$, the limit $\lim_{n \to \infty} \langle x_n^*, d \rangle$ exists.

The following statement was shown in [4, Lemma 2].

LEMMA 2.2. *If $\psi : \mathbb{R}^k \to \mathbb{R}$ is semismooth at $\bar{x} \in \mathbb{R}^k$, then the directional derivative $\psi'(\bar{x}; d)$ exists for all $d \in \mathbb{R}^k$ and equals the limit $\lim_{n \to \infty} \langle x_n^*, d \rangle$, where x_n^* is any of the sequences from Definition* 2.1.

The concept of semismoothness may be carried over to sets via the Euclidean distance function d^e.

DEFINITION 2.3. A set $A \subseteq \mathbb{R}^k$ is called semismooth at $\bar{x} \in \operatorname{cl} A$ if for any sequence $x_n \to \bar{x}$ with $x_n \in A$ and $\|x_n - \bar{x}\|^{-1}(x_n - \bar{x}) \to d$ it holds that $\langle x_n^*, d \rangle \to 0$ for all selections $x_n^* \in \partial^c d_A^e(x_n)$.

PROPOSITION 2.4. *If $A \subseteq \mathbb{R}^k$ is closed and d_A^e is semismooth at $\bar{x} \in A$, then A is semismooth at $\bar{x}$.*

Proof. Let x_n, x_n^* be arbitrary sequences in Definition 2.3. Taking $t_n := \|x_n - \bar{x}\|$ and $d_n := t_n^{-1}(x_n - \bar{x})$ in Definition 2.1, we derive from Lemma 2.2 the existence of the directional derivative $d_A^{e\prime}(\bar{x}; d)$ as well as

$$\langle x_n^*, d \rangle \to d_A^{e\prime}(\bar{x}; d) = \lim_{t \downarrow 0, d' \to d} t^{-1}\big(d_A^e(\bar{x} + td') - d_A^e(\bar{x})\big)$$

$$= \lim_{n \to \infty} t_n^{-1}\big(d_A^e(x_n) - d_A^e(\bar{x})\big) = 0,$$

where the representation of the directional derivative relies on d_A^e being Lipschitz. ∎

3. CHARACTERIZATION OF CALMNESS

We start with the main result of this paper.

THEOREM 3.1. *Consider the multifunction $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ given by*

$$M(y) := \{x \in \Omega \mid g(x) + y \in \Lambda\}, \tag{3}$$

where $\Omega \subseteq \mathbb{R}^p$ *and* $\Lambda \subseteq \mathbb{R}^m$ *are closed subsets and* $g : \mathbb{R}^p \to \mathbb{R}^m$ *is locally Lipschitz near some* $\bar{x}$ *with* $(0, \bar{x}) \in \operatorname{Gph} M$. *Let the following assumptions hold*:

1. Ω *is regular at* $\bar{x}$.
2. *One of the following two conditions holds true*:
   a. *There is some norm* $\|\cdot\|_+$ *on* $\mathbb{R}^m$ *such that the value function* $\pi(x) := \min_{z \in \Lambda} \|z - g(x)\|_+$ *is semismooth at* $\bar{x}$.
   b. Ω *is semismooth at* $\bar{x}$.
3. *The constraint qualification*

$$\bigcup_{y^* \in N_\Lambda(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(y^*) \cap (-\operatorname{bd} N_\Omega(\bar{x})) = \emptyset, \tag{4}$$

holds true.

Then M *is calm at* $(0, \bar{x})$.

REMARK 3.2. Alternatively, Eq. (4) can be written as

$$\left.\begin{array}{r} D^*g(\bar{x})(y^*) \cap -\operatorname{bd} N_\Omega(\bar{x}) \neq \emptyset \\ y^* \in N_\Lambda(g(\bar{x})) \end{array}\right\} \Rightarrow y^* = 0.$$

Proof. Assume by contradiction that M is not calm at $(0, \bar{x})$. By definition, there exist sequences $x_n \to \bar{x}$, $x_n \in M(y_n)$, $y_n \to 0$, such that $d(x_n, M(0)) > nd(0, y_n)$, where the distance on the right side is generated by $\|\cdot\|_+$. Hence

$$d(x_n, M(0)) > nd\big(0, M^{-1}(x_n)\big). \tag{5}$$

In particular, $x_n \in M(y_n)$ implies that $x_n \in \Omega$. Clearly, $d(x_n, M(0)) > 0$ for all n because of (5). Further, for the function defined in assumption 2 in Theorem 3.1, we have

$$\pi(x) = d(g(x), \Lambda) = d(0, -g(x) + \Lambda) = d\big(0, M^{-1}(x)\big) \qquad \forall x \in \Omega. \tag{6}$$

We observe that $\pi(x_n) > 0$, because otherwise $x_n \in M(0)$, in contrast to the foregoing statement. Consequently, each x_n is an ε-minimizer of the function $\pi + \delta_\Omega$ with $\varepsilon := \pi(x_n)$. [Recall that $\bar{x} \in M(0)$, hence $\bar{x} \in \Omega$ and $\inf(\pi + \delta_\Omega) = \pi(\bar{x}) = 0$.] Because $\pi + \delta_\Omega$ is a proper lower-semicontinuous function, the application of Ekeland's variational principle (with ε as earlier and $\lambda := (n/2)\pi(x_n)$) yields for each $n \in \mathbb{N}$ the existence of a point $\tilde{x}_n$ such that

$$\pi(\tilde{x}_n) + \delta_\Omega(\tilde{x}_n) \leq \pi(x_n) + \delta_\Omega(x_n) \tag{7}$$

$$\|\tilde{x}_n - x_n\| \leq (n/2)\pi(x_n) \tag{8}$$

$$\tilde{x}_n \in \arg\min\{\pi(x) + (2/n)\|\tilde{x}_n - x\| \,|\, x \in \Omega\}. \tag{9}$$

Note that (7) implies that $\pi(\tilde{x}_n) + \delta_\Omega(\tilde{x}_n) \leq \pi(x_n)$ because $x_n \in \Omega$, and hence $\delta_\Omega(\tilde{x}_n) = 0$. Consequently, $\tilde{x}_n \in \Omega$, and the formulation of (9) is justified. From (8), (6), and (5), we infer that $\tilde{x}_n \to \bar{x}$ and $\pi(\tilde{x}_n) > 0$. Indeed, $\pi(\tilde{x}_n) = 0$ would imply the contradiction to (5),

$$d(x_n, M(0)) \leq \|\tilde{x}_n - x_n\| \leq (n/2)\pi(x_n) = (n/2)d(0, M^{-1}(x_n)).$$

Applying the necessary optimality conditions to (9), we deduce that

$$0 \in \partial\pi(\tilde{x}_n) + N_\Omega(\tilde{x}_n) + (2/n)\mathbb{B}^*.$$

Hence there exist sequences $u_n^* \in \partial\pi(\tilde{x}_n)$ and $v_n^* \in -N_\Omega(\tilde{x}_n)$ such that $\|u_n^* - v_n^*\|^* \leq 2/n$ for all $n \in \mathbb{N}$. Because π is Lipschitz near $\bar{x}$, the sequence $\{u_n^*\}$ is bounded. Consequently, because of the last relation, $\{v_n^*\}$ must be bounded too. By extracting appropriate subsequences, one arrives at

$$u_{n'}^* \to u^* \in \partial\pi(\bar{x}) \qquad \text{and} \qquad v_{n'}^* \to u^* \in -N_\Omega(\bar{x}) \tag{10}$$

by virtue of the multifunctions $\partial\pi(\cdot)$ and $N_\Omega(\cdot)$ having closed graphs.

Next, we verify the relation

$$u^* \in \{D^*g(\bar{x})(y^*) | y^* \in N_\Lambda(g(\bar{x})) \setminus \{0\}\}. \tag{11}$$

Toward this aim, denote $\sigma(x, z) := \|g(x) - z\|_+$ and $A(x) := \{z \in \Lambda | \pi(x) = \sigma(x, z)\}$. Because $\Lambda \neq \emptyset$ [because $g(\bar{x}) \in \Lambda$], $A(x) \neq \emptyset$ for each $x \in \mathbb{R}^p$. Furthermore, well-known results from parametric optimization (e.g., [9, Corollary 7.42]) imply that Gph A is closed and A is uniformly bounded around each $x \in \mathbb{R}^p$. This, along with the fact that σ is locally Lipschitz, allows to apply Theorem 4.1 in [5] to the function π. One gets the inclusion

$$\partial\pi(\tilde{x}_n) \subseteq \bigcup_{y^* \in \mathbb{R}^m,\, z \in A(\tilde{x}_n)} \{x_1^* + x_2^* \in \mathbb{R}^p | x_1^* \in D^*Q(\tilde{x}_n, z)(y^*),$$
$$(x_2^*, y^*) \in \partial\sigma(\tilde{x}_n, z)\},$$

where $Q:\mathbb{R}^p \rightrightarrows \mathbb{R}^m$ denotes the constant multifunction $Q(x) := \Lambda \ \forall x \in \mathbb{R}^p$. Clearly $\operatorname{Gph} Q = \mathbb{R}^p \times \Lambda$, $N_{\operatorname{Gph} Q}(\tilde{x}_n, z) = \{0\} \times N_\Lambda(z)$, and the definition of the coderivative implies that

$$D^*Q(\tilde{x}_n, z)(y^*) = \begin{cases} 0 & \text{if } y^* \in -N_\Lambda(z) \\ \emptyset & \text{else} \end{cases}.$$

Consequently,

$$\partial\pi(\tilde{x}_n) \subseteq \bigcup_{y^* \in -N_\Lambda(z),\, z \in A(\tilde{x}_n)} \{x^* \in \mathbb{R}^p | (x^*, y^*) \in \partial\sigma(\tilde{x}_n, z)\}. \quad (12)$$

Putting $f(x, z) := g(x) - z$, we have that $\sigma = \|\cdot\|_+ \circ f$, and the chain rule for Lipschitz mappings in [7, Corollary 5.8] yields that

$$\partial\sigma(\tilde{x}_n, z) \subseteq \bigcup_{s \in \partial\|\cdot\|_+(g(\tilde{x}_n) - z)} \partial\langle s, f\rangle(\tilde{x}_n, z). \quad (13)$$

Because $\langle s, f\rangle(x, z) = s^T g(x) - s^T z$, the sum rule in [7, Corollary 4.6] provides that

$$\begin{aligned} \partial\langle s, f\rangle(\tilde{x}_n, z) &= [\partial\langle s, g\rangle(\tilde{x}_n) \times \{0\}] + [\{0\} \times \{-s\}] \\ &= \partial\langle s, g\rangle(\tilde{x}_n) \times \{-s\}. \end{aligned}$$

Furthermore, because $\pi(\tilde{x}_n) > 0$ implies that $f(\tilde{x}_n, z) \neq 0$ for all $z \in A(\tilde{x}_n)$, we derive from convex analysis that

$$\partial\|\cdot\|_+(g(\tilde{x}_n) - z) = \{s \in \mathbb{S}^* | \langle s, g(\tilde{x}_n) - z\rangle = \|g(\tilde{x}_n) - z\|_+\} \quad (z \in A(\tilde{x}_n)),$$

where $\mathbb{S}^*$ denotes the unit sphere in $\mathbb{R}^m$ equipped with the dual norm of $\|\cdot\|_+$. Combining the previous relations with (13) gives

$$\partial\sigma(\tilde{x}_n, z) \subseteq \{D^*g(\tilde{x}_n)(s) \times \{-s\} | s \in \mathbb{S}^*\} \quad (z \in A(\tilde{x}_n)),$$

where we used the relation $D^*g(\tilde{x}_n)(s) = \partial\langle s, g\rangle(\tilde{x}_n)$ which is valid for Lipschitzian mappings (cf. [7, Proposition 4.6]. Inserting the last inclusion into (12) gives

$$\partial\pi(\tilde{x}_n) \subseteq \bigcup_{z \in A(\tilde{x}_n)} \{D^*g(\tilde{x}_n)(s) | s \in \mathbb{S}^* \cap N_\Lambda(z)\},$$

which holds for all $n \in \mathbb{N}$, because n was arbitrarily fixed. Therefore, along with the sequence $u^*_{n'}$ defined in (10), we have sequences $s_{n'}$ and $z_{n'}$ such that

$$u^*_{n'} \in D^*g(\tilde{x}_{n'})(s_{n'}), \qquad s_{n'} \in \mathbb{S}^* \cap N_\Lambda(z_{n'}), \qquad \text{and} \qquad z_{n'} \in A(\tilde{x}_{n'}).$$

Because $\tilde{x}_{n'} \to \bar{x}$ and A is uniformly bounded around $\bar{x}$, we may extract subsequences $\{s''_n\}$ and $\{z''_n\}$ such that $s_{n''} \to \bar{s} \in \mathbb{S}^*$ and $z_{n''} \to \bar{z}$. By closedness of Gph A (see remark above), it follows that $\bar{z} \in A(\bar{x}) = \{g(\bar{x})\}$, because $g(\bar{x}) \in \Lambda$. Furthermore, $(u^*_{n''}, -s_{n''}) \in N_{\text{Gph}\, g}(\tilde{x}_{n''}, g(\tilde{x}_{n''}))$ according to the definition of the coderivative. Because the graph of the normal cone mapping, $N_{\text{Gph}\, g}$, is closed, we infer that $\bar{s} \in N_\Lambda(g(\bar{x}))$ and, by (10),

$$(u^*_{n''}, -s_{n''}) \to (u^*, -\bar{s}) \in N_{\text{Gph}\, g}(\bar{x}, g(\bar{x})).$$

Consequently, $u^* \in D^*g(\bar{x})(\bar{s})$ with $\bar{s} \in \mathbb{S}^* \cap N_\Lambda(g(\bar{x}))$, which eventually implies (11).

Now (5), (6), and (8) yield $\|x_n - \bar{x}\| > n\pi(x_n) \geq (n/2)\pi(x_n) \geq \|\tilde{x}_n - x_n\|$. Taking into account the already obtained relations $\pi(\bar{x}) = 0 < \pi(\tilde{x}_n) \leq \pi(x_n)$ [see (7)], we arrive at

$$0 < \frac{\pi(\tilde{x}_n) - \pi(\bar{x})}{\|\tilde{x}_n - \bar{x}\|} \leq \frac{\pi(x_n)}{\|x_n - \bar{x}\| - \|\tilde{x}_n - x_n\|} \leq \frac{2}{n}. \tag{14}$$

From the sequence $\tilde{x}_{n'}$ [corresponding to $u^*_{n'}$ in (10)], we extract a subsequence $\tilde{x}_{n^*}$ (recall that $\tilde{x}_{n'} \neq \bar{x}$) such that

$$\lim_{n^* \to \infty} \frac{\tilde{x}_{n^*} - \bar{x}}{\|\tilde{x}_{n^*} - \bar{x}\|} = h \quad \text{for some} \quad h \in \mathbb{R}^p, \text{with } \|h\| = 1. \tag{15}$$

Clearly, $h \in K_\Omega(\bar{x}) = T^c_\Omega(\bar{x})$ by assumption 1 in Theorem 3.1. Because $u^*_{n^*} \in \partial\pi(\tilde{x}_{n^*})$ and $v^*_{n^*} \in -N_\Omega(\tilde{x}_{n^*})$ [see the derivation on top of (10)], the trivial representation

$$\tilde{x}_{n^*} = \bar{x} + \lambda_{n^*} h_{n^*},$$

with $\qquad \lambda_{n^*} := \|\tilde{x}_{n^*} - \bar{x}\| \downarrow 0 \qquad$ and $\qquad h_{n^*} := \dfrac{\tilde{x}_{n^*} - \bar{x}}{\|\tilde{x}_{n^*} - \bar{x}\|} \to h,$

provides that $u^*_{n^*} \in \partial\pi(\bar{x} + \lambda_{n^*} h_{n^*})$ and $v^*_{n^*} \in -N_\Omega(\bar{x} + \lambda_{n^*} h_{n^*})$. Under assumption 2a in Theorem 3.1, Lemma 2.2 gives

$$\langle u^*, h \rangle = \lim_{n^* \to \infty} \langle u^*_{n^*}, h \rangle = \pi'(\bar{x}; h). \tag{16}$$

With π being locally Lipschitz and using (14), we can write its directional derivative as

$$\pi'(\bar{x};h) = \lim_{t\downarrow 0, h'\to h} \frac{\pi(\bar{x}+th') - \pi(\bar{x})}{t}$$
$$= \lim_{n^*\to\infty} \frac{\pi(\bar{x}+\lambda_{n^*}h_{n^*}) - \pi(\bar{x})}{\lambda_{n^*}} = 0, \qquad (17)$$

whence $\langle u^*, h\rangle = 0$. We verify the same relation under assumption 2b; it is evident in the case where $u^* = 0$, so let $u^* \neq 0$. From (10), it follows that $v^*_{n^*} \neq 0$ for n^* large enough. Next, we refer to the identity

$$N_\Omega(\tilde{x}_{n^*}) \cap \mathbb{B}_2 = \partial d^e_\Omega(\tilde{x}_{n^*}) \qquad (18)$$

(see Ex. 8.53 in [9]). Consequently, with $\tilde{v}^*_{n^*} := -v^*_{n^*}/\|v^*_{n^*}\|_2$, we obtain $\tilde{v}^*_{n^*} \in \partial d^e_\Omega(\tilde{x}_{n^*}) \subseteq \partial^c d^e_\Omega(\tilde{x}_{n^*})$. Assumption 2b then yields

$$\|u^*\|^{-1}\langle u^*, h\rangle = -\lim_{n^*\to\infty}\langle \tilde{v}^*_{n^*}, h\rangle = 0,$$

whence again $\langle u^*, h\rangle = 0$.

Using that $\langle u^*, h\rangle = 0$ under either assumption 2a or 2b, we get for arbitrarily small $\varepsilon > 0$ that $\langle u^* - \varepsilon h, h\rangle = -\varepsilon < 0$. Because $h \in T^c_\Omega(\bar{x})$, this implies that $u^* - \varepsilon h \notin -N^c_\Omega(\bar{x})$. On the other hand, $u^* \in -N^c_\Omega(\bar{x})$, according to (10). Consequently, $u^* \in \mathrm{bd}(-N^c_\Omega(\bar{x})) = -\mathrm{bd}\, N^c_\Omega(\bar{x}) = -\mathrm{bd}\, N_\Omega(\bar{x})$ by the regularity assumption 1, which together with (11) provides a contradiction to (4). ∎

Note that as a result of the regularity assumption 1, we may replace N^c_Ω by N_Ω in the constraint qualification (4). The obtained result may be illustrated in one dimension as follows.

EXAMPLE 3.3. In Theorem 3.1, let $\Lambda := \mathbb{R}_-$, $g(x) := x$ and $\Omega := \mathbb{R}_+$. Then the multifunction M in (3) is easily verified to be calm at the point $(0,0)$ of its graph. Clearly, assumptions 1 and 2 (actually both, 2a and 2b) are satisfied. Furthermore, the constraint qualification (4) reduces to the condition $\nabla g(0)(=1) \notin -\mathrm{bd}\, N^c_\Omega(0)(=\{0\})$, which is certainly satisfied. On the other hand, we have $\nabla g(0) \in -N_\Omega(0)(= -N^c_\Omega(0) = \mathbb{R}_+)$, so that the criterion (2), designed for the stronger Aubin property, does not apply.

The following example illustrates that the regularity of Ω in assumption 1 and Corollary 4.1 cannot be dispensed with in general, so the constraint qualification (4) is not sufficient for calmness in case of arbitrary closed sets Ω.

EXAMPLE 3.4. In the context of Theorem 3.1, define $\Lambda := \mathbb{R}_-$, $g(x) := x^2$, and $\Omega := \{n^{-1/2} | n \in \mathbb{N}\} \cup \{0\}$. Then Ω is closed but is not regular at $\bar{x} := 0$. Obviously, $M(0) = \{0\}$; hence $(0, \bar{x}) \in \operatorname{Gph} M$. Furthermore, assumption 2a is satisfied, because

$$\pi(x) = \min_{z \leq 0} |z - g(x)| = \max\{g(x), 0\}$$

is semismooth as the maximum of two semismooth functions [4, Theorem 5]. Finally, we can easily verify that $T^c_\Omega(\bar{x}) = \{0\}$; hence $N^c_\Omega(\bar{x}) = \mathbb{R}$. Consequently, $-\operatorname{bd} N^c_\Omega(\bar{x}) = \emptyset$, so assumption 3 is trivially fulfilled. On the other hand, M is not calm at $(0, \bar{x})$. (Take sequences $x_n := n^{-1/2}$ and $y_n := -n^{-1}$ for establishing a contradiction to the definition of calmness.)

Recall that regularity and semismoothness of a set Ω are completely independent properties (assumptions 1 and 2b).

EXAMPLE 3.5. Let $\Omega := \operatorname{epi}(-|x|) \subseteq \mathbb{R}^2$. Then

$$d^e_\Omega(x, y) = \max\left\{0, \min\left\{(x - y)/\sqrt{2}, -(x + y)/\sqrt{2}\right\}\right\}$$

is semismooth as a min-max composition of semismooth functions (cf. [4]). Invoking Proposition 2.4, we see that Ω is a semismooth set that clearly fails to be regular at $(0, 0)$. Conversely, define

$$\Omega := \bigcup_{n \in \mathbb{N}} \left\{(x, y) \in \mathbb{R}^2 | x \leq -n^{-1}, 0 \leq y \leq n^{-2}\right\}.$$

Calculating $K_\Omega(0, 0) = T^c_\Omega(0, 0) = \mathbb{R}_-$, we verify that Ω is regular at $(\bar{x}, \bar{y}) = (0, 0)$. On the other hand, taking the sequence $(x_n, y_n) := (-n^{-1}, n^{-2})$, we get $(x_n, y_n) \in \Omega$ and

$$\|(x_n, y_n) - (\bar{x}, \bar{y})\|^{-1}((x_n, y_n) - (\bar{x}, \bar{y})) \to d := (-1, 0).$$

Because $(1, 0) \in \partial^c d^e_\Omega(x_n, y_n)$, the contradiction $\langle z^*_n, d\rangle = -1$ to Definition 2.3 follows with $z^*_n \equiv (1, 0)$. Hence Ω is not semismooth at $(\bar{x}, \bar{y})$.

In the rest of this section we identify structures of the abstract constraint set Ω that render superfluous all technical assumptions of Theorem 3.1 such that the constraint qualification (4) becomes the only condition of calmness for M. First, we indicate a situation where Ω satisfies assumptions 1 and 2b. Toward this aim, let a set A be described by the following system of inequalities:

$$A = \left\{x \in \mathbb{R}^k | f_i(x) \leq 0, i = 1, \dots, l\right\}. \tag{19}$$

Further, for $x \in A$, let $I(x)$ denote the standard set of active inequalities,

$$I(x) := \left\{i \in \{1, 2, \dots, l\} | f_i(x) = 0\right\}.$$

LEMMA 3.6. *Let A be given as in* (19), *where the* $f_i : R^k \to R$ *are continuous. If at some* $\bar{x} \in A$, *the* $f_i (i \in I(\bar{x}))$ *are locally Lipschitzian, regular, and semismooth, and if moreover, the constraint qualification*

$$0 \notin \operatorname{conv}\{\partial^c f_i(\bar{x}) | i \in I(\bar{x})\} \tag{20}$$

is satisfied, then A is regular and semismooth at $\bar{x}$.

Proof. We may assume that $I(\bar{x}) \neq \emptyset$, because otherwise the assertion is trivial. Defining

$$f(x) := \max\{f_i(x) | i \in I(\bar{x})\},$$

we have that, because of the continuity of the f_i, the set A is locally described around $\bar{x}$ by the inequality $f(x) \le 0$, where also $f(\bar{x}) = 0$. Obviously, f is locally Lipschitzian, regular and semismooth as a maximum of functions with these properties (see Proposition 2.3.12 in [2] for regularity and the beginning of the proof of Corollary 4.1 for semismoothness of the maximum of semismooth functions). Consequently, A is regular at $\bar{x}$ (see [2], Corollary 2, p. 56). Concerning the semismoothness, consider arbitrary x_n, x_n^*, and d as in Definition 2.3, hence $x_n \to \bar{x}$, $x_n \in A$, $\|x_n - \bar{x}\|^{-1}(x_n - \bar{x}) \to d$, and

$$x_n^* \in \partial^c d_A^e(x_n) = N_A^c(x_n) \cap \mathbb{B}_2, \tag{21}$$

as a consequence of (18), where the index "c" can be appended to the subdifferential and normal cone in view of the regularity of A. Clearly, if $f(x_n) < 0$, then $N_A^c(x_n) = \{0\}$. Hence if $f(x_n) < 0$ for all n large enough, then $x_n^* = 0$ and $\langle x_n^*, d\rangle \to 0$ trivially implies the condition of semismoothness. Therefore, we may assume that $f(x_{n_k}) = 0$ for some subsequence. In virtue of the mean value theorem for the Clarke subdifferential (see Theorem 2.3.7 in [2]), there exists a sequence $\tau_{n_k} \in [0, 1]$ such that

$$\begin{aligned} 0 &= f(\bar{x}) - f(x_{n_k}) = \langle z_{n_k}^*, \bar{x} - x_{n_k}\rangle \\ z_{n_k}^* &\in \partial^c f\big(\bar{x} + \tau_{n_k}(x_{n_k} - \bar{x})\big) = \partial^c f\big(\bar{x} + t_{n_k}\|x_{n_k} - \bar{x}\|^{-1}(x_{n_k} - \bar{x})\big) \\ t_{n_k} &= \tau_{n_k}\|x_{n_k} - \bar{x}\|. \end{aligned} \tag{22}$$

On noting that $t_{n_k} \to 0$ due to boundedness of τ_{n_k} and that $\|x_{n_k} - \bar{x}\|^{-1}(x_{n_k} - \bar{x}) \to d$, the semismoothness f at $\bar{x}$ guarantees according to Lemma 2.2 that the directional derivative $f'(\bar{x}; d)$ exists and equals zero:

$$f'(\bar{x}; d) = \lim_k \langle z_{n_k}^*, d\rangle = \lim_k \langle z_{n_k}^*, \|x_{n_k} - \bar{x}\|^{-1}(x_{n_k} - \bar{x})\rangle = 0. \tag{23}$$

Here the second equality relies on f being locally Lipschitzian—which implies boundedness of the sequence $z^*_{n_k}$—while the last equality is from (22). On the other hand, it generally holds that $N^c_A(x_n) \subseteq \bigcup_{\lambda \geq 0} \partial^c f(x_n)$ [2, Corollary 1], hence, according to (21), there exist sequences $\lambda_n \geq 0$ and $u^*_n \in \partial^c f(x_n)$ such that $x^*_n = \lambda_n u^*_n$. Now our constraint qualification (20) is equivalent to the condition $0 \notin \partial^c f(\bar{x})$ (see Theorem 2.8.2 in [2]). Because the Clarke subdifferential is convex, it can be strictly separated from 0, and the half-space containing $\partial^c f(\bar{x})$ will contain all subdifferentials $\partial^c f(x_n)$ for n large enough because of uppersemicontinuity of the subdifferential mapping. This implies the existence of some $c > 0$ such that $\|u^*_n\| \geq c$ for n large enough. Then from (21) we get the boundedness of λ_n,

$$\lambda_n = \|u^*_n\|^{-1} \|x^*_n\| \leq c^{-1}.$$

Exploiting the semismoothness of f at $\bar{x}$ a second time (now with $t_n = \|x_n - \bar{x}\|$ and $d_n = \|x_n - \bar{x}\|^{-1}(x_n - \bar{x}) \to d$), we deduce from (16) that $\langle u^*_n, d \rangle \to f'(\bar{x}; d) = 0$ and finally arrive at

$$\langle x^*_n, d \rangle = \lambda_n \langle u^*_n, d \rangle \to 0$$

by virtue of boundedness of the λ_n. Summarizing, A is semismooth at $\bar{x}$. ∎

Consequently, the result of Theorem 3.1 simplifies as follows for common structures of the abstract constraints.

COROLLARY 3.7. *In the setting of Theorem* 3.1, *let* Ω *be convex or described by a finite number of regular and semismooth Lipschitz inequalities as in* (19) *that satisfy the regularity condition* (20). *Then the constraint qualification* (4) *implies calmness of* M *at* $(0, \bar{x})$.

Proof. In both cases, Ω is a regular and semismooth set. For the second case, this was shown in Lemma 3.6. For convex Ω, regularity is clear, and semismoothness follows from semismoothness of the convex distance function d^e_Ω [4, Proposition 3] via Proposition 2.4. ∎

Another relevant instance of abstract sets Ω that allow direct application of the criterion (4) without further technical assumptions is given by unions of smooth inequalities [in contrast to intersections as in (19)]. At the same time, this structure reflects a situation where our criterion (4) coincides with condition (2) ensuring the Aubin property. Note that in general, as pointed out by Example 3.3, both conditions differ significantly.

DEFINITION 3.8. We call $A \subseteq \mathbb{R}^k$ a "disjunctive set" if there exists a continuously differentiable mapping $f: \mathbb{R}^k \to \mathbb{R}^l$ for some $l \in \mathbb{N}$ such that

$$A = \left\{x \in \mathbb{R}^k | \exists i \in \{1, \dots l\} : f_i(x) \leq 0\right\} \text{ and}$$

$$\operatorname{rank}\left\{\nabla f_i(x) | i \in J(x)\right\} = \sharp J(x) \ \forall x \in A,$$

where $J(x) = \{i \in \{1, \dots l\} | f_i(x) = \min_{j \in \{1,\dots,l\}} f_j(x) = 0\}$ denotes a modified set of active indices.

Similar to conventional sets of active indices, the continuity of f implies that $J(x) \subseteq J(\bar{x})$ for x close to $\bar{x}$ ($x, \bar{x} \in A$).

PROPOSITION 3.9. *Let $A \subseteq \mathbb{R}^k$ be a disjunctive set. Then*

$$N_A(\bar{x}) = \mathbb{R}_+ \cdot \bigcup_{i \in J(\bar{x})} \{\nabla f_i(\bar{x})\} \tag{24}$$

for all $\bar{x} \in A$.

Proof. The assertion is obvious for the cases $\sharp J(\bar{x}) \leq 1$, because then either $J(\bar{x}) = \emptyset$ (hence $\bar{x} \in \operatorname{int} A$ and $N_A(\bar{x}) = \{0\}$) or $J(\bar{x}) = \{i^*\}$ for some $i^* \in \{1, \dots l\}$. Then, locally around $\bar{x}$, the set A is described by the single continuously differentiable inequality $f_{i^*}(x) \leq 0$ with, according to Definition 3.8, $\nabla f_{i^*}(\bar{x}) \neq 0$; hence $N_A(\bar{x}) = \mathbb{R}_+ \cdot \nabla f_{i^*}(\bar{x})$. Now let $\sharp J(\bar{x}) \geq 2$ and, without loss of generality, assume that $\{1, 2\} \subseteq J(\bar{x})$; hence $f_1(\bar{x}) = f_2(\bar{x}) = 0$. Because $\nabla f_1(\bar{x})$ and $\nabla f_2(\bar{x})$ are linearly independent according to Definition 3.8, there exists some d with $\langle \nabla f_2(\bar{x}), d \rangle \leq 0$ and $\langle \nabla f_1(\bar{x}), d \rangle > 0$. Clearly, for the polar to the contingent cone introduced in Section 2, it holds that $\hat{N}_A(\bar{x}) \subseteq \mathbb{R}_+ \cdot \nabla f_1(\bar{x})$ because $\{x | f_1(x) \leq 0\} \subseteq A$. On the other hand, the first of the preceding inequalities ensures that d belongs to the contingent cone of the set $\{x | f_2(x) \leq 0\} \subseteq A$ at $\bar{x}$; hence d belongs to $K_A(\bar{x})$. Then, by the second of the preceding inequalities, $\nabla f_1(\bar{x}) \notin \hat{N}_A(\bar{x})$. Summarizing, we arrive at $\hat{N}_A(\bar{x}) = \{0\}$.

Now let $x \in A$ be close to but different from $\bar{x}$. In each of the cases $\sharp J(x) = 0, 1$, the cones $\hat{N}_A(x)$, $N_A(x)$ coincide, and we have $\hat{N}_A(x) = \{0\}$ or $\hat{N}_A(x) = \mathbb{R}_+ \cdot \nabla f_{i^*}(x)$, respectively, for some $i^* \in J(\bar{x})$ according to the foregoing remarks related to $\bar{x}$ rather than x. If instead $\sharp J(x) \geq 2$, then $\hat{N}_A(x) = \{0\}$ (again according to the foregoing remarks related to $\bar{x}$ rather than x). Recalling how N_A is generated from $\hat{N}_A$ (see Sec. 2), this altogether yields the inclusion "$\subseteq$" in (24). For the reverse inclusion, it suffices to show that $\nabla f_i(\bar{x}) \in N_A(\bar{x})$ for all $i \in J(\bar{x})$. This, however, follows again from the full rank condition in Definition 3.8, which ensures for each $i \in J(\bar{x})$ the existence of some sequence $x_n \to \bar{x}$ such that $f_i(x_n) = 0$ and $f_j(x_n) > 0$ for $j \in J(\bar{x}) \setminus \{i\}$. Then $J(x_n) = \{i\}$ and (see above) $\hat{N}_A(x) = \mathbb{R}_+ \cdot \nabla f_i(x_n)$, so $0 \neq \nabla f_i(x_n) \in \hat{N}_A(x)$. Passing to the limit $n \to \infty$, we obtain the desired relation $\nabla f_i(\bar{x}) \in N_A(\bar{x})$ by continuous differentiability of f. ∎

COROLLARY 3.10. *Let $A \subseteq \mathbb{R}^k$ be a disjunctive set and $k > 1$. Then $N_A(\bar{x}) = \operatorname{bd} N_A(\bar{x})$ for all $\bar{x} \in A$.*

This corollary confirms the coincidence of Mordukhovich's criterion and our criterion [(2) and (4), respectively] for disjunctive sets Ω, so there is no chance to distinguish algebraically between calmness and Aubin property in this situation. As a simple example, take $\bar{x} = 0$ and $\Omega = \mathrm{epi}(-|x|)$, which is a disjunctive set with $f_1(x) = x$, $f_2(x) = -x$. Then, formally, Theorem 3.1 cannot be applied because assumption 1 is violated. Nevertheless, (4) can be invoked by virtue of its coincidence with (2) according to Corollary 3.10.

4. APPLICATION TO OPTIMIZATION AND NONLINEAR COMPLEMENTARITY PROBLEMS

With particular choices of Λ, we can specialize the results of Section 3 for various constraint mappings arising in applications. The simplest case corresponds to standard mathematical programs with inequality constraints, where $\Lambda = \mathbb{R}^m_-$ and, consequently,

$$N_\Lambda(g(\bar{x})) = \{y^* \in \mathbb{R}^m | y_i^* \geq 0 \text{ for } i \in I(\bar{x}) \text{ and } y_i^* = 0 \text{ otherwise}\}, \quad (25)$$

with $I(\bar{x}) := \{i \in \{1, \dots, m\} | g_i(\bar{x}) = 0\}$ being the set of active indices.

COROLLARY 4.1. *Consider the multifunction M in* (3) *with $\Lambda = \mathbb{R}^m_-$ at a point* $(0, \bar{x}) \in \mathrm{Gph}\, M$. *Assume that*

1. *Ω is regular at $\bar{x}$.*
2. *One of the following two conditions holds true*:
   a *All components g_i of g are semismooth at $\bar{x}$.*
   b *Ω is semismooth at $\bar{x}$.*
3. *The following constraint qualification holds true*:

$$\sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) \cap (-\mathrm{bd}\, N_\Omega(\bar{x})) = \emptyset \qquad \text{for all } \lambda_i \geq 0 \text{ with } \sum_{i \in I(\bar{x})} \lambda_i = 1. \quad (26)$$

Then M is calm at $(0, \bar{x})$.

Proof. Choosing $\|\cdot\|_+$ as the l_∞ norm, the specific structure $\Lambda = \mathbb{R}^m_-$ considered here provides for π in Theorem 3.1

$$\pi(x) = \max_{1,\dots,m} [g_i(x)]_+ = \max\{g^1(x), \dots, g^m(x), 0\}. \quad (27)$$

Now the last expression is a composition of the semismooth function $\max\{\cdot, \dots, \cdot\}$ with a mapping with semismooth components under assumption 2a. Applying Theorem 5 in [4], we derive the semismoothness of π at

$\bar{x}$. Obviously, it suffices now to check assumption 3 of Theorem 3.1. Assuming its violation, there would exist some u^* with

$$u^* \in \bigcup_{v^* \in N_\Lambda(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(v^*) \cap (-\text{bd}\, N_\Omega(\bar{x})).$$

In particular, according to (25), there exists some $v^* \in \mathbb{R}^m_+ \setminus \{0\}$ such that $v_i^* = 0$ for $i \notin I(\bar{x})$ and

$$u^* \in D^*g(\bar{x})(v^*) = \partial\Bigg(\sum_{i\in I(\bar{x})} v_i^* g_i\Bigg)(\bar{x}) \subseteq \sum_{i\in I(\bar{x})} v_i^* \partial g_i(\bar{x}),$$

where we used that $\partial(\lambda f) = \lambda\partial(f)$ for $\lambda \geq 0$ and the sum rule $\partial(f_1 + f_2) \subseteq \partial(f_1) + \partial(f_2)$ for locally Lipschitzian functions. Because $v^* \neq 0$ and $-\text{bd}\, N_\Omega^c(\bar{x})$ is a cone, we have $c^{-1}u^* \in -\text{bd}\, N_\Omega^c(\bar{x})$ for $c := \sum_{i\in I(\bar{x})} v_i^* > 0$ as well as

$$c^{-1}u^* \in \sum_{i\in I(\bar{x})} (c^{-1}v_i^*) g_i(\bar{x}).$$

This contradicts assumption 3 of Corollary 4.1, and hence assumption 3 of Theorem 3.1 has to be satisfied. ∎

Clearly, the technical assumptions 1 and 2 of Corollary 4.1 can be circumvented in special situations in the same way as described in Section 3 with respect to Theorem 3.1. Hence, constraint qualification (26) is automatically sufficient for calmness of M in the following cases:

- Ω is convex or described by a finite number of regular and semi-smooth inequalities as in (19) that satisfy the regularity condition (20).
- Ω is a disjunctive set as in Definition 3.8.
- Ω is a regular set and g is convex or continuously differentiable or a maximum or minimum over a continuously and compactly indexed family of continuously differentiable functions (in all of which cases g becomes semismooth; see [4]).

In the completely regular case (Ω and g regular), the constraint qualification (26) can be weakened by passing to the boundary a second time. Note that no semismoothness assumption is required here.

THEOREM 4.2. *Consider the multifunction M in* (3) *with* $\Lambda = \mathbb{R}^m_-$ *at a point* $(0, \bar{x}) \in \text{Gph}\, M$. *Assume that*

1. *Ω is regular at $\bar{x}$.*
2. *All components g_i are regular at $\bar{x}$.*

3. *The following constraint qualification holds true*:

$$\left(\operatorname{bd}\left\{\sum_{i\in I(\bar{x})}\lambda_i\partial g_i(\bar{x})\,|\,\lambda_i\geq 0(i\in I(\bar{x})),\ \sum_{i\in I(\bar{x})}\lambda_i=1\right\}\right)\cap(-\operatorname{bd}N_\Omega(\bar{x}))$$
$$=\emptyset.$$

Then M is calm at $(0,\bar{x})$.

Proof. We define π as in (27) and introduce the function

$$\rho(x)=\max\{g^1(x),\dots,g^m(x)\},$$

which is regular according to assumption 2 of Theorem 4.2. If $\rho(\bar{x})<0$ [i.e., $g(\bar{x})\in\operatorname{int}\Lambda$], then the continuity of g entails calmness of M at $(0,\bar{x})$ in a trivial way. Hence let $\rho(\bar{x})=\pi(\bar{x})=0$. Then, because of the regularity in assumption 2, we have (see [2], Theorem 2.8.2).

$$\partial^c\rho(\bar{x})=\left\{\sum_{i\in I(\bar{x})}\lambda_i\partial g_i(\bar{x})\,|\,\lambda_i\geq 0(i\in I(\bar{x})),\ \sum_{i\in I(\bar{x})}\lambda_i=1\right\}. \quad (28)$$

Suppose now that M fails to be calm at $(0,\bar{x})$. Repeating the proof of Theorem 3.1 until (10), one gets sequences $\tilde{x}_n, u_n^*$ such that $u_n^*\in\partial\pi(\tilde{x}_n)$, $\pi(\tilde{x}_n)>0$, and $u_n^*\to u^*$ (omitting indices of subsequences). By definition, $\pi(\tilde{x}_n)>0$ implies that π and ρ coincide in a neighborhood of $\tilde{x}_n$. Consequently, $\partial\pi(\tilde{x}_n)=\partial\rho(\tilde{x}_n)=\partial^c\rho(\tilde{x}_n)$, and we arrive at (by upper semicontinuity of Clarke's subdifferential)

$$u^*\in\partial^c\rho(\bar{x}). \quad (29)$$

The second relation in (10) may also be written as (due to regularity of Ω)

$$u^*\in -N_\Omega^c(\bar{x}). \quad (30)$$

Finally, (14) may be rewritten in terms of ρ as

$$0<\frac{\rho(\tilde{x}_n)-\rho(\bar{x})}{\|\tilde{x}_n-\bar{x}\|}<\frac{2}{n}.$$

Similar to (17), this entails that

$$\rho'(\bar{x};h)=0 \text{ for } h \text{ given by } (15). \quad (31)$$

Note that for all these relations derived from the proof of Theorem 3.1, no semismoothness arguments were used. Now (30) and (31) give $\langle u^*,h\rangle\geq 0$.

On the other hand, with ρ being regular, its directional derivative coincides with its generalized directional derivative in the sense of Clarke and hence can be represented as

$$\rho'(\bar{x}; h) = \max\{\langle u^{*\prime}, h\rangle | u^{*\prime} \in \partial^c\rho(\bar{x})\}. \tag{32}$$

Then (29) and (31) provide that $\langle u^*, h\rangle \leq 0$, whence $\langle u^*, h\rangle = 0$. Combining this last relation with (30), it follows that $u^* \in -\mathrm{bd}\, N^c_\Omega(\bar{x})$ with the same argumentation as in the last paragraph of the proof of Theorem 3.1. Finally, the orthogonality of u^* and h along with (32) also entail that

$$u^* \in \operatorname{argmax}\{\langle u^{*\prime}, h\rangle | u^{*\prime} \in \partial^c\rho(\bar{x})\}.$$

Because the linear function $\langle h, \cdot\rangle$ necessarily attains its maximum on $\partial^c\rho(\bar{x})$ at a boundary point, it results that

$$u^* \in \mathrm{bd}\, \partial^c\rho(\bar{x}) \cap -\mathrm{bd}\, N^c_\Omega(\bar{x})$$

which, in view of (28), is a contradiction to assumption 3 of Theorem 4.2. ∎

In the trivial case of a single inequality $g(x) \leq 0$ (without abstract constraints), the constraint qualification in Theorem 4.2 turns into the condition $0 \notin \mathrm{bd}\, \partial g(\bar{x})$. Of course, in the smooth case with a single inequality, this amounts to the condition $\nabla g(\bar{x}) \neq 0$, which is sufficient even for the stronger Aubin property of the constraint set mapping. A substantial gain over the criterion $0 \notin \partial g(\bar{x})$ (sufficient for the Aubin property) therefore occurs either in a nonsmooth setting (as, for instance, in the simple convex example discussed in Section 1), or in a smooth setting with several inequalities. To illustrate it, consider the situation, where $\Omega = \mathbb{R}^n$. Then the constraint qualification from Theorem 4.2 attains the form

$$0 \notin \mathrm{bd}\big(\mathrm{conv}\{\nabla g_i(\bar{x}) | i \in I(\bar{x})\}\big), \tag{33}$$

whereas the classical Mangasarian–Fromovitz constraint qualification, written in "dual" form, reads

$$0 \notin \mathrm{conv}\{\nabla g_i(\bar{x}) | i \in I(\bar{x})\}. \tag{34}$$

Note that (33) is much weaker than (34) and yet ensures calmness of the constraint mapping, hence guarantees existence of Lagrange multipliers for local minima.

Combining the last remarks with those following Corollary 3.10 and with Example 3.3, we have identified several significant circumstances—independently for the set Ω and for the function g—under which the criteria (4) and (2) differ or coincide.

We now give an example that highlights the necessity of the additional regularity assumption on g in Theorem 4.2:

EXAMPLE 4.3. In the context of Theorem 4.2, let $m = 2$ and define $g_1(x) := x^2$, $g_2(x) := -|x|$, $\Omega := \mathbb{R}$, $\bar{x} := 0$. Then $M(0) = \{0\}$; hence $(0, \bar{x}) \in \text{Gph } M$. Obviously, assumption 1 is satisfied. Furthermore, we have

$$\left(\text{bd}\left\{\sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) | \lambda_i \geq 0 (i \in I(\bar{x})), \sum_{i\in I(\bar{x})} \lambda_i = 1\right\}\right) \cap (-\text{bd } N_\Omega(\bar{x}))$$
$$= (\text{bd}\{\lambda_1 \cdot \{0\} + \lambda_2 \cdot \{-1, 1\} | \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\}) \cap (-\text{bd}\{0\})$$
$$= (\text{bd}[-1, 1]) \cap \{0\} = \emptyset.$$

This entails assumption 3. Hence, all assumptions of Theorem 4.2 with the exception of the regularity of g_2 are satisfied. Now, with the sequences $x_n = n^{-1}$ and $y_n = (-n^{-2}, 0)$, it is easily checked that M fails to be calm at $\bar{x}$. On the other hand, the weaker result of Corollary 4.1 still applies in the sense that (26) is violated.

Consider now a situation associated with a parametric nonlinear complementarity problem (NCP). For a given $p \in \mathbb{R}^k$, find $x \in \mathbb{R}^n_+$ such that

$$f(p, x) \geq 0, \qquad \langle x, f(p, x)\rangle = 0, \qquad (p, x) \in \Omega, \tag{35}$$

where $f : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuously differentiable and $\Omega \subseteq \mathbb{R}^k \times \mathbb{R}^n$ is closed. Putting $b(p, x) := (x, -f(p, x))$, the nonabstract part of (35) can be equivalently written in the form $b(p, x) \in \text{Gph } N_{\mathbb{R}^n_+}$, in which case N reduces to the classical normal cone of convex analysis. We define a multifunction $M : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^k \times \mathbb{R}^n$ by

$$M(y) = \{(p, x) \in \Omega | b(p, x) + y \in \text{Gph } N_{\mathbb{R}^n_+}\}.$$

THEOREM 4.4. *Let* $(0, \bar{p}, \bar{x}) \in \text{Gph } M$ *and assume that*

1. Ω *is closed and regular at* $(\bar{p}, \bar{x})$.
2. *The constraint qualification*

$$\left.\begin{array}{l}\left(-[\nabla_p f(\bar{p}, \bar{x})]^T z, w - [\nabla_x f(\bar{p}, \bar{x})]^T z\right) \in -\text{bd } N^c_\Omega(\bar{p}, \bar{x}) \\ \text{for some } (w, z) \in N_{\text{Gph } N_{\mathbb{R}^n_+}}(\bar{x}, -f(\bar{p}, \bar{x}))\end{array}\right\}$$
$$\Rightarrow w = 0, z = 0$$

is satisfied.

Then M is calm at $(0, \bar{p}, \bar{x})$.

Proof. Our aim is to apply Theorem 3.1 with

$$m := 2n,\ p := k + n,\ g := b,\ \Lambda := \operatorname{Gph} N_{\mathbb{R}^n_+}.$$

Endowing the space $\mathbb{R}^n \times \mathbb{R}^n$ with the norm

$$\|(v_1, v_2)\|_+ := \sqrt{\sum_{i=1}^{n} \left(\max\{|v_1^i|, |v_2^i|\}\right)^2},$$

the following point-to-set distance has been calculated in [8] (Proposition 5.1):

$$\operatorname{dist}\left(0, -b(p, x) + \operatorname{Gph} N_{\mathbb{R}^n_+}\right) = \|\min\{x, F(p, x)\}\|_2,$$

where the minimum must be understood componentwise. But the left side is exactly the value function π of assumption 2(a) in Theorem 3.1. Because concave functions (like “min”) and convex functions (like $\|\cdot\|_2$) are semismooth, π itself is semismooth as a composition.

Finally, observing that

$$D^*b(\bar{p}, \bar{x})((w, z) = \begin{bmatrix} 0 & -\left[\nabla_p f(\bar{p}, \bar{x})\right]^T \\ \mathbf{I}_n & -\left[\nabla_x f(\bar{p}, \bar{x})\right]^T \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix},$$

we verify that assumption 2 of Theorem 4.4 entails assumption 3 of Theorem 3.1. Summarizing, assumptions 1, 2(a), and 3 of Theorem 3.1 are satisfied. ∎

5. CONCLUSION

In a number of perturbed equilibrium problems, including the foregoing NCP, the map M attains the form

$$M(y) = \{(p, x) \in \Omega | b(p, x) + y \in \operatorname{Gph} Q\}, \tag{36}$$

where $b : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^{2n}$ is continuously differentiable and $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^{2n}$ is a multifunction with the closed graph. In this situation the presented theory can be applied, provided that we endow $\mathbb{R}^{2n}$ with a suitable norm $\|\cdot\|_+$ such that the value function

$$\pi(p, x) := \min_{z \in \operatorname{Gph} Q} \|z - b(p, x)\|_+$$

satisfies the requirements of Theorem 3.1. The choice of this norm depends naturally on the structure of the (possibly complicated) nonconvex set Gph Q.

Consider now the optimization problem

$$\text{minimize } \vartheta(x) \quad \text{subject to} \quad x \in M(0) \cap \Omega, \tag{37}$$

where $\vartheta : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitzian objective, and assume that $\hat{x}$ is its local solution. By virtue of [10] and Lemma 3.1, under the assumptions of Theorem 3.1 there exists a real $R > 0$ and neighborhood $\mathscr{U}$ of $\hat{x}$ such that $\hat{x}$ solves the penalized problem

$$\text{minimize } \vartheta(x) + R\pi(x) \quad \text{subject to} \quad x \in \Omega \cap \mathscr{U}. \tag{38}$$

Function π thus can be used as a penalty for the numerical solution of (37). Moreover, on the basis of (38), we can derive necessary optimality conditions for (37).

ACKNOWLEDGMENT

The authors thank A. Jourani (Université de Bourgogne, Dijon) for intensive discussion on the subject of this paper, which led to an improved version of Theorem 4.2. They extend their gratitude to B. Mordukhovich (Wayne State University, Detroit), B. Kummer (Humboldt University, Berlin) and to an anonymous referee for useful comments and suggestions.

REFERENCES

1. F. H. Clarke, A new approach to Lagrange multipliers, *Math. Oper. Res.* **1** (1976), 165–174.
2. F. H. Clarke, "Optimization and Nonsmooth Analysis," Wiley, New York, 1983.
3. A. D. Ioffe, Necessary and sufficient conditions for a local minimum, Part I, *SIAM J. Cont. Optimiz.* **17** (1979), 245–265.
4. R. Mifflin, Semismooth and semiconvex functions in constrained optimization, *SIAM J. Cont. Optimiz.* **15** (1977), 959–972.
5. B. S. Mordukhovich, Sensitivity analysis in nonsmooth optimization, *SIAM Proc. Appl. Math.* **58** (1992), 32–46.
6. B. S. Mordukhovich, Complete characterization of openness, metric regularity and Lipschitzian properties of multifunctions, *Trans. Amer. Math. Soc.* **340** (1993), 1–35.
7. B. S. Mordukhovich, Generalized differential calculus for nonsmooth and set-valued mappings, *J. Math. Anal. Appl.* **183** (1994), 250–288.
8. J. V. Outrata, On mathematical programs with complementarity constraints, *Optimiz. Meth. Software* **14** (2000), 117–137.
9. R. T. Rockafellar and R. J.-B. Wets, "Variational Analysis," Springer, Berlin, 1997.
10. J. J. Ye and X. Y. Ye, Necessary optimality conditions for optimization problems with variational inequality constraints, *Math. Oper. Res.* **22** (1997), 977–997.