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Sufficient Conditions for Error Bounds and Applications*

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Abstract. Our aim in this paper is to present sufficient conditions for error bounds in terms of Fréchet and limiting Fréchet subdifferentials in general Banach spaces. This allows us to develop sufficient conditions in terms of the approximate subdifferential for systems of the form $(x, y) \in C \times D$, $g(x, y, u) = 0$, where g takes values in an infinite-dimensional space and u plays the role of a parameter. This symmetric structure offers us the choice of imposing conditions either on C or D. We use these results to prove the nonemptiness and weak-star compactness of Fritz–John and Karush–Kuhn–Tucker multiplier sets, to establish the Lipschitz continuity of the value function and to compute its subdifferential and finally to obtain results on local controllability in control problems of nonconvex unbounded differential inclusions.

Key Words. Error bounds, Sufficient condition, Sensitivity analysis, Local controllability.

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1. Introduction

Consider an inequality system

$$f(x, u) \leq 0, \tag{1}$$

where f is a given extended real-valued function. It is a familiar consideration in mathematics to seek to solve this inequality for x, while viewing u as a parameter. Typically this is done in a neighborhood of a given point $(\bar{x}, \bar{u})$ for which (1) is satisfied, and the important issues are these: For a given u near $\bar{u}$, does there continue to be at least one value of x for which (1) holds? How does this set $S(u)$ of solutions vary with u?

One outcome is to consider the following metric inequalities in some neighborhood of $(\bar{x}, \bar{u})$:

$$d(x, S(u)) \leq a \max(0, f(x, u))$$

for some constant $a > 0$. These inequalities are called *error bounds* for system (1).

Error bounds of constraint systems on compact sets are closely related to the concept of *metric regularity*: A set-valued mapping $S\colon U \rightrightarrows X$ with $(\bar{u}, \bar{x}) \in \operatorname{gph} S$ is said to be metrically regular at $\bar{u}$ for $\bar{x}$ if there exists a constant $\kappa > 0$ such that

$$d(u, S^{-1}(x)) \leq \kappa d(x, S(u))$$

for all (u, x) in some neighborhood of $(\bar{u}, \bar{x})$. The infimum of κ for which the above inequality holds is the *modulus of metric regularity* and is denoted by $\operatorname{reg} S(\bar{u} \mid \bar{x})$. Probably, the first regularity result goes back to Graves (see [7]), stating that a continuously differentiable mapping S between Banach spaces, having a surjective differential $DS(\bar{u})$ at $\bar{u}$, is metrically regular at $(\bar{u}, S(\bar{u}))$. For recent surveys on metric regularity, see [5], [6], [10] and [30].

The primary objective of this paper is to develop sufficient conditions for error bounds and to give applications of the results obtained to optimization problems and sensitivity analysis as well as controllability in control problems of nonconvex unbounded differential inclusions. There are several conditions ensuring these error bounds. These conditions are in general expressed in terms of subdifferentials or axiomatic subdifferentials (see [16], [2], [3], [19]–[23], [32], [12], [29] and references therein). Some of these subdifferentials depend on the data space. For example, Fréchet subdifferentials and limiting Fréchet subdifferentials characterize Asplund spaces (see [27] and [28]). Sufficient conditions given before in terms of these two subdifferentials are formulated only in Asplund spaces.

Our aim here is to give sufficient conditions in general Banach spaces for error bounds in terms of Fréchet and limiting Fréchet subdifferentials which are the smallest ones among all subdifferentials or axiomatic subdifferentials. This allows us to obtain sufficient conditions for general systems in terms of the approximate subdifferential by Ioffe [8], [9].

The rest of the paper is organized as follows. Section 2 contains basic definitions. Section 3 is devoted to the study of local and global error bounds related to system (1) and to the system

$$x \in C \quad \text{and} \quad g(x, u) \in D,$$

where g takes values in a finite-dimensional space. The conditions presented in this section are given only in terms of Fréchet and limiting Fréchet subdifferentials. Based on the results in Section 3, we develop in Section 4 sufficient conditions in terms of the approximate subdifferentials for error bounds for systems of the form

$$(x, y) \in C \times D \quad \text{and} \quad g(x, y, u) = 0,$$

where g takes values in an infinite-dimensional space. This symmetric structure offers us the choice of imposing conditions either on C or D to get error bounds for this system. As a particular case of these systems we consider systems of the form

$$x \in C, \qquad g(x) \in D,$$

since they can be transformed into the form $(x, y) \in C \times D$, $\quad g(x) - y = 0$, where g takes values in a Banach space. In Section 5 we give some applications of our results. We prove the nonemptiness and weak-star compactness of Fritz–John and Karush–Kuhn–Tucker multiplier sets, establish the Lipschitz continuity of the value function and compute its subdifferential and finally obtain results on local controllability in control problems of nonconvex unbounded differential inclusions.

2. Notation and Preliminaries

In order to make the paper as short as possible, some definitions and the complete wording of the results are not repeated here, and the reader is referred to [24]–[26], [8] and [9] if required. Throughout we assume that X, Y and Z are Banach spaces endowed with some norm denoted by $\|\cdot\|$ to which we associate the distance function $d(\cdot, C)$ to a set C. We also assume that (U, d) is a metric space. $B(x, r)$ refers to the ball centered at x and of radius r.

We write $x \xrightarrow{f} x_0$, and $x \xrightarrow{S} x_0$ to express $x \to x_0$ with $f(x) \to f(x_0)$ and $x \to x_0$ with $x \in S$, respectively.

Let f be an extended real-valued function on $X \times U$. The partial limiting Fréchet subdifferential of f at (x_0, u_0) in x with respect to u is the set

$$\partial_x^{\mathrm{F}} f(x_0, u_0) = w^*\text{-seq-} \limsup_{\substack{(x,u) \xrightarrow{f} (x_0, u_0) \\ \varepsilon \to 0^+}} \partial_x^{\varepsilon} f(x, u),$$

where

$$\partial_x^{\varepsilon} f(x, u) = \left\{ x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h, u) - f(x, u) - \langle x^*, h \rangle}{\|h\|} \geq -\varepsilon \right\}$$

is the partial ε-Fréchet subdifferential of f at (x, u). When f depend only on x we denote it by $\partial^{\mathrm{F}} f(x)$.

The limiting Fréchet normal cone to a closed set $S \subset X$ at a point $x \in S$ is given by

$$N_{\mathrm{F}}(S, x) = \partial^{\mathrm{F}} \delta_S(x),$$

where δ_S denotes the indicator function of S. The basic theory of the Fréchet and limiting Fréchet subdifferentials, with fairly comprehensive references and remarks, is developed in the paper by Mordukhovich and Shao, see [28].

If f is an extended real-valued function on X, we write for any subset S of X,

$$f_S(x) = \begin{cases} f(x) & \text{if} \quad x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

The function

$$d^- f(x,h) = \liminf_{\substack{u\to h\\ t\downarrow 0}} t^{-1}(f(x+tu) - f(x))$$

is the lower Dini directional derivative of f at x and the Dini ε-subdifferential of f at x is the set

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le d^- f(x;h) + \varepsilon\|h\|, \forall h \in X\}$$

for $x \in \operatorname{Dom} f$ and $\partial_\varepsilon^- f(x) = \emptyset$ if $x \notin \operatorname{Dom} f$, where $\operatorname{Dom} f$ denotes the effective domain of f. For $\varepsilon = 0$ we write $\partial^- f(x)$.

By $\mathcal{F}(X)$ we denote the collection of finite-dimensional subspaces of X. The approximate subdifferentials of f at $x_0 \in \operatorname{Dom} f$ is defined by the following expressions (see [8] and [9]):

$$\partial_{\mathrm{A}} f(x_0) = \bigcap_{L\in\mathcal{F}(X)} \limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \bigcap_{L\in\mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_o\\ \varepsilon\downarrow 0}} \partial_\varepsilon^- f_{x+L}(x),$$

where

$$\limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \{x^* \in X^* : x^* = w^*\text{-}\lim x_i^*, x_i^* \in \partial^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0\},$$

that is, the set of w^*-limits of all such nets.

The G-normal cone to a closed set $C \subset X$ at x_0 is defined by

$$N_G(C, x_0) = \mathbb{R}_+ \, \partial_{\mathrm{A}} d(C, x_0).$$

The limiting Fréchet and approximate subdifferentials are both infinite-dimensional extensions of the nonconvex subdifferential introduced in [24].

Using the remark following Proposition 1.6 and Proposition 2.4 in [14] we obtain the following result.

Proposition 1. *Let v: $X \mapsto \mathbb{R}$ be a function which is locally Lipschitzian at $\bar{x}$ with Lipschitz constant k_v. Then the following are equivalent*:

(i) $x^* \in \partial_{\mathrm{A}} v(\bar{x})$.
(ii) $(x^*, -1) \in N_G(\text{graph } v; (\bar{x}, v(\bar{x})))$.
(iii) $(x^*, -1) \in (k_v + 1)\partial_{\mathrm{A}} d(\text{graph } v; (\bar{x}, v(\bar{x})))$.
(iv) *For all $L \in \mathcal{F}(X)$ there are nets $x_i^* \to x^*$, $x_i \to \bar{x}$, $\varepsilon_i \to 0^+$ and $r_i \to 0^+$ such that*

$$\|x_i^*\| \le (k_v + 1)(1 + \varepsilon_i),$$

$$v(x) - v(x_i) - \langle x_i^*, x - x_i\rangle + \varepsilon_i\|x - x_i\| \ge 0, \qquad \forall x \in B(x_i, r_i) \cap (L + x_i).$$

Finally we recall that the mapping g: $X \times U \mapsto Y$ is of *class $\mathcal{C}^1$ at $(\bar{x}, \bar{u})$ in x with respect to u* if g and its partial derivative $D_x g(x,u)$ are continuous at $(\bar{x}, \bar{u})$.

3. Error Bounds Using Fréchet Subdifferentials

It is well known that some Banach spaces may be characterized in terms of some subdifferentials. For example the Dini subdifferential characterizes the Weak Trustworthy spaces. The ε-Fréchet (and limiting Fréchet) subdifferential gives a characterization of Asplund spaces. To give sufficient conditions for error bounds for systems in terms of the limiting Fréchet subdifferential, the previous works assume that the space is Asplund (see the papers by Mordukhovich and Shao [27], [28]). Our aim here is to obtain these results in general Banach spaces.

Here we consider the following systems:

$$f(x,u) \le 0 \qquad (\mathbf{S}_1)$$

and

$$x \in C \quad \text{and} \quad g(x,u) \in D, \qquad (\mathbf{S}_2)$$

where $f\colon X \times U \mapsto \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, C and D are closed sets in X and $\mathbb{R}^m$, and $g\colon X \times U \mapsto \mathbb{R}^m$ is a mapping. Here $\mathbb{R}^m$ is endowed with the euclidean norm which will also be denoted by $\|\cdot\|$.

The corresponding parametric solution set is defined by the multivalued mapping

$$S_1(u) = \{x \in X : f(x,u) \le 0\}$$

and

$$S_2(u) = \{x \in C : g(x,u) \in D\}.$$

We begin with system ($\mathbf{S}_1$) for which we give a sufficient condition ensuring a local error bound. We set

$$B_f((x,u),r) := \{(x',u') \in B(x,r) \times B(u,r) : |f(x',u') - f(x,u)| \le r\}.$$

Theorem 2. *Suppose $f(\bar{x},\bar{u}) = 0$ and there exists $r > 0$ such that*

$$\forall (x,u) \in B_f((\bar{x},\bar{u}),r), \quad x \notin S_1(u), \qquad \forall \varepsilon \in]0,r[, \quad 0 \notin \partial_x^{\varepsilon} f(x,u).$$

Then there exist constants $a > 0$, $b > 0$ and $s > 0$ such that

$$d(x, S_1(u)) \le a d(f(x,u), \mathbb{R}_-)$$

for all $x \in B(\bar{x},s)$, $u \in B(\bar{u},s)$, with $f(x,u) \le b$.

Proof. Suppose the contrary. Then there exist sequences $x_n \to \bar{x}$ and $u_n \to \bar{u}$ such that

$$d(x_n, S_1(u_n)) > n d(f(x_n,u_n), \mathbb{R}_-) \quad \text{and} \quad f(x_n,u_n) \le \frac{1}{n}. \qquad (2)$$

Note that $x_n \notin S_1(u_n)$ or equivalently $f(x_n, u_n) > 0$. Set $\varepsilon_n^2 = f(x_n, u_n)$, $\lambda_n = \min(n\varepsilon_n^2, \varepsilon_n)$ and $s_n = \varepsilon_n^2/\lambda_n$. It is easy to see that $\varepsilon_n, \lambda_n, s_n \to 0^+$. Consider the function $h(x) = d(f(x, u_n), \mathbb{R}_-)$. Then

$$h(x_n) \leq \inf_{x \in X} h(x) + \varepsilon_n^2.$$

By the lower semicontinuity of h, Ekeland's variational principle ensures the existence of $x_n' \in X$ satisfying

$$\|x_n' - x_n\| \leq \lambda_n, \tag{3}$$

$$h(x_n') \leq h(x) + s_n\|x_n' - x\|, \qquad \forall x \in X. \tag{4}$$

Note that by (2) and the definition of λ_n we have $d(x_n, S_1(u_n)) > \lambda_n$ and then by (3) we obtain $x_n' \notin S_1(u_n)$. Since f is lower semicontinous, $h(x)$ coincides with $f(x, u_n)$ in a neighborhood of x_n' ($\mathcal{V}(x_n')$) and hence by (4) we get for some subsequence $(x'_{m(n)})$ that

$$f(x'_{m(n)}, u_{m(n)}) \leq f(x, u_{m(n)}) + s_{m(n)}\|x'_{m(n)} - x\|, \qquad \forall x \in \mathcal{V}(x_n'),$$

and then $f(x'_{m(n)}, u_{m(n)}) \to f(\bar{x}, \bar{u})$ and

$$0 \in \partial_x^{s_{m(n)}} f(x'_{m(n)}, u_{m(n)})$$

and this contradicts our assumption. □

Remark 3. In the paper by Lewis and Pang [17], they gave conditions for the existence of a global error bound for a convex inequality system and established a necessary and sufficient condition for a closed convex set defined by a closed proper function to possess a global error bound in terms of a natural residual.

We have the following corollary of Theorem 2.

Corollary 4. *Suppose that* $f(\bar{x}, \bar{u}) = 0$ *and that*

$$0 \notin \partial_x^{\mathrm{F}} f(\bar{x}, \bar{u}).$$

Then the conclusion of Theorem 2 *holds.*

As the limiting Fréchet subdifferential is always included in Clarke's one our result implies all those expressed in terms of Clarke's subdifferential. Take for example the real-valued function f defined on $\mathbb{R}^3$ by $f(x, y, u) = |x| - |y|$. Then

$$S(u) = \{(x, y) \in \mathbb{R}^2 : |x| \leq |y|\}$$

and at the point $(0, 0)$ we have $0 \notin \partial^{\mathrm{F}}_{(x,y)} f(0, 0, 0)$ while 0 is in Clarke's subdifferential of f at $(0, 0)$.

We continue with system ($\mathbf{S}_1$) in which we assume that $f(x,u) = f(x)$. We give a condition for which a global error bound holds. The proof is similar to the previous one.

Theorem 5. *Suppose that the solution set S_1 of system ($\mathbf{S}_1$) is nonempty and there exists $r > 0$ such that*

$$\forall x \notin S_1, \quad \forall \varepsilon \in]0, r[, \qquad 0 \notin \partial^{\varepsilon} f(x). \tag{5}$$

Then there exists a constant $a > 0$ such that

$$d(x, S_1) \leq a d(f(x), \mathbb{R}_-), \qquad \forall x \in X.$$

Remark 6. In the case where f is convex, the relation (5) always implies

$$\forall x \notin S_1, \qquad 0 \notin \partial f(x). \tag{6}$$

However, the converse does not hold. To see this, it is enough to consider the function

$$f(x) = \begin{cases} x^2 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad -1 \leq x \leq 0, \\ (x+1)^2 & \text{if} \quad x < -1. \end{cases}$$

This example also shows that (6) is not sufficient to get an error bound, even when we have S_1 compact.

Now we pass to system ($\mathbf{S}_2$). The following result is a consequence of Theorem 2.

Theorem 7. *Suppose that*:

(i) *$(\bar{x}, \bar{u})$ is a solution of system* ($\mathbf{S}_2$).
(ii) *g is of class $\mathcal{C}^1$ at $(\bar{x}, \bar{u})$ in x with respect to u ,i.e. g and its partial derivative $D_x g(x,u)$ are continuous at $(\bar{x}, \bar{u})$.*

Then $(\beta) \Longrightarrow (\alpha)$, where

(α) *there exists $a > 0$ and $r > 0$ such that*

$$d(x, S_2(u)) \leq a d(g(x,u), D)$$

for all $x \in C \cap B(\bar{x}, r)$ and all $u \in B(\bar{u}, r)$; and

(β) *there is no $y^* \in N_{\mathrm{F}}(D, g(\bar{x}, \bar{u}))$, $y^* \neq 0$, satisfying $0 \in y^* \circ D_x g(\bar{x}, \bar{u}) + N_{\mathrm{F}}(C, \bar{x})$.*

Proof. Consider the function $f \colon X \times U \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x,u) = \begin{cases} d(g(x,u), D) & \text{if} \quad x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$S_2(u) = \{x \in X : f(x,u) \leq 0\}.$$

With these definitions and taking into account the continuity of g in both variables x and u, assertion (α) is equivalent to the conclusion of Theorem 2. Suppose that (α) is false. Then, by Theorem 2, there are sequences $x_n \to \bar{x}$, with $x_n \in C$, $u_n \to \bar{u}$ and $\varepsilon_n \to 0^+$ such that

$$x_n \notin S_2(u_n) \quad \text{and} \quad 0 \in \partial_x^{\varepsilon_n} f(x_n, u_n). \tag{7}$$

So there exists $r_n \to 0^+$ such that

$$f(x_n, u_n) \leq f(x, u_n) + 2\varepsilon_n \|x_n - x\|, \qquad \forall x \in B(x_n, r_n),$$

or equivalently

$$d(g(x_n, u_n), D) \leq d(g(x, u_n), D) + 2\varepsilon_n \|x_n - x\|, \qquad \forall x \in B(x_n, r_n) \cap C. \tag{8}$$

Let $d_n \in D$ such that

$$d(g(x_n, u_n), D) = \|g(x_n, u_n) - d_n\|.$$

Then $d_n \to g(\bar{x}, \bar{u})$ and by (8) we obtain

$$\|g(x_n, u_n) - d_n\| \leq \|g(x, u_n) - d_n\| + 2\varepsilon_n \|x - x_n\|, \qquad \forall x \in B(x_n, r_n) \cap C,$$

and

$$\|g(x_n, u_n) - d_n\| \leq \|g(x_n, u_n) - y\|, \qquad \forall y \in D.$$

Set $y_n^* = (g(x_n, u_n) - d_n)/\|g(x_n, u_n) - d_n\|$. Using the euclidean structure of $\mathbb{R}^m$ and the fact that g is of class $\mathcal{C}^1$ at $(\bar{x}, \bar{u})$ in x with respect to u we get a sequence $s_n \to 0^+$ such that

$$-y_n^* \circ D_x g(x_n, u_n) \in N_{\mathrm{F}}^{s_n}(C, x_n)$$

and

$$y_n^* \in N_{\mathrm{F}}^{s_n}(D, d_n).$$

Extracting a subsequence if necessary we may assume that $y_n^* \to y^*$, with $\|y^*\| = 1$ (because the space has a finite dimension). Thus there exists $y^* \in N_{\mathrm{F}}(D, g(\bar{x}, \bar{u}))$, $y^* \neq 0$, such that $0 \in y^* \circ D_x g(\bar{x}, \bar{u}) + N_{\mathrm{F}}(C, \bar{x})$. However, this inclusion contradicts assertion (β). □

4. Error Bounds Using Approximate Subdifferentials

In this section we consider parametrized systems of the form

$$(x, y) \in C \times D \quad \text{and} \quad g(x, y, u) = 0, \tag{$\mathbf{S}_3$}$$

where C and D are closed sets in X and Y and $g\colon X \times Y \times U \mapsto Z$ is a mapping. Our system may be nonlinear with respect to the perturbation u. Let $S_3(u)$ be the set of solutions to system $(\mathbf{S}_3)$. Before stating the following theorem, we recall the following notion by Borwein and Strojwas [1]. A set $S \subset X$ is said to be *compactly epi-Lipschitzian* at $x_0 \in S$ if there exist $\gamma > 0$ and a norm compact set $H \subset X$ such that

$$S \cap B(x_0, \gamma) + B(0, t\gamma) \subset S - tH \qquad \text{for all} \quad t \in]0, \gamma[.$$

Theorem 8. *Suppose that*:

(i) $(\bar{x}, \bar{y}, \bar{u})$ *is a solution of system* $(\mathbf{S}_3)$.
(ii) g *is of class* $\mathcal{C}^1$ *at* $(\bar{x}, \bar{y}, \bar{u})$ *in* x *with respect to* (y, u) *with surjective partial derivative* $D_x g(\bar{x}, \bar{y}, \bar{u})$.
(iii) g *is of class* $\mathcal{C}^1$ *at* $(\bar{x}, \bar{y}, \bar{u})$ *in* y *with respect to* (x, u) *with partial derivative* $D_y g(\bar{x}, \bar{y}, \bar{u})$.
(iv) C *is compactly epi-Lipschitzian at* $\bar{x}$.

Then $(\beta) \Longrightarrow (\alpha)$, *where*

(α) *there exist* $a > 0$ *and* $r > 0$ *such that*

$$d((x, y), S_3(u)) \leq a\|g(x, y, u)\|$$

for all $x \in C \cap B(\bar{x}, r)$, $y \in D \cap B(\bar{y}, r)$ *and* $u \in B(\bar{u}, r)$; *and*
(β) *there is no* $z^* \in Z^*$, $z^* \neq 0$, *satisfying*

$$z^* \circ D_x g(\bar{x}, \bar{y}, \bar{u}) \in k_g \partial_{\mathrm{A}} d(C, \bar{x}), \qquad z^* \circ D_y g(\bar{x}, \bar{y}, \bar{u}) \in k_g \partial_{\mathrm{A}} d(D, \bar{y}),$$

where k_g *is a Lipschitz constant of* g *at* $(\bar{x}, \bar{y}, \bar{u})$.

Proof. Consider the function $f\colon X \times Y \times U \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x, y, u) = \begin{cases} \|g(x, y, u)\| & \text{if} \quad (x, y) \in C \times D, \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$S_3(u) = \{(x, y) \in X \times Y : f(x, y, u) \leq 0\}.$$

Suppose that (α) is false. Then, as in the proof of Theorem 7, there are sequences $((x_n, y_n)) \subset C \times D$, $(u_n) \subset U$ and $(r_n), (s_n) \subset \mathbb{R}_+$, with $(x_n, y_n) \to (\bar{x}, \bar{y})$, $u_n \to \bar{u}$, $r_n \to 0^+$ and $s_n \to 0^+$, such that

$$g(x_n, y_n, u_n) \neq 0$$

and

$$\|g(x_n, y_n, u_n)\| \leq \|g(x, y, u_n)\| + s_n\|(x - x_n, y - y_n)\|$$

for all $(x, y) \in (C \times D) \cap B((x_n, y_n), r_n)$. Thus, there exists $z_n^* \in Z^*$, with $\|z_n^*\| = 1$, such that

$$z_n^* \circ D_x g(x_n, y_n, u_n) \in (k_g + s_n)\partial_{\mathrm{A}} d(x_n, C) + s_n B^*$$

and

$$z_n^* \circ D_y g(x_n, y_n, u_n) \in (k_g + s_n)\partial_{\mathrm{A}} d(y_n, D) + s_n B^*.$$

Now using the surjectivity of $D_x g(\bar{x}, \bar{y}, \bar{u})$ and the fact that g is of class $\mathcal{C}^1$ there exists $r > 0$, not depending on $n \geq n_0$, such that

$$\|z_n^* \circ D_x g(x_n, y_n, u_n)\| \geq r.$$

Extracting a subnet we may assume that $z_n^* \to z^*$ with respect to the weak*-topology, with $z^* \circ D_x g(\bar{x}, \bar{y}, \bar{u}) \in k_g \partial_{\mathrm{A}} d(\bar{x}, C)$ and $z^* \circ D_y g(\bar{x}, \bar{y}, \bar{u}) \in k_g \partial_{\mathrm{A}} d(\bar{y}, D)$. Since C is compactly epi-Lipschitzian at $\bar{x}$, then by Lemma 2.3 in [15] there exist $h_1, \dots, h_k \in X$, not depending on n, such that

$$r \leq \max_{i=1,\dots,k} \langle z_n^* \circ D_x g(x_n, y_n, u_n), h_i \rangle$$

and hence

$$r \leq \max_{i=1,\dots,k} \langle z^* \circ D_x g(\bar{x}, \bar{y}, \bar{u}), h_i \rangle.$$

Thus $z^* \neq 0$ and this contradiction completes the proof. □

As a particular case of the previous system, we consider systems of the form

$$(x, y) \in C \times D \quad \text{and} \quad g_1(x) - g_2(y) = 0, \qquad (\mathbf{S}_4)$$

where C and D are closed sets in X and Y, respectively, and $g_1\colon X \to Z$ and $g_2\colon Y \to Z$ are mappings. Let $S_4(z) := \{(x, y) \in C \times D : \ g_1(x) - g_2(y) = z\}$.

Corollary 9. *Suppose that*:

(i) $(\bar{x}, \bar{y})$ *is a solution of system* $(\mathbf{S}_4)$.
(ii) g_1 *is of class* $\mathcal{C}^1$ *at* $\bar{x}$ *with surjective derivative* $Dg_1(\bar{x})$.
(iii) g_2 *is of class* $\mathcal{C}^1$ *at* $\bar{y}$ *with derivative* $Dg_2(\bar{y})$.
(iv) C *is compactly epi-Lipschitzian at* $\bar{x}$.

Then $(\beta) \Longrightarrow (\alpha)$, *where*

(α) *there exist* $a > 0$ *and* $r > 0$ *such that*

$$d((x, y), S_4(z)) \leq a\|g_1(x) - g_2(y) + z\|$$

for all $x \in C \cap B(\bar{x}, r)$, $y \in D \cap B(\bar{y}, r)$ *and* $z \in B(0, r)$; *and*

(β) *there is no* $z^* \in Z^*$, $z^* \neq 0$, *satsfying*

$$-z^* \circ Dg_1(\bar{x}) \in k_g \partial_A d(C, \bar{x}), \qquad z^* \circ Dg_2(\bar{x}) \in k_g \partial_A d(D, \bar{y}),$$

where k_g *is a Lipschitz constant of* $g := g_1 - g_2$ *at* $(\bar{x}, \bar{y})$.

The following corollary generalizes in the differentiable case the result by Jourani and Thibault [15] in which it is assumed that D is compactly epi-Lipschitzian at $g(\bar{x})$. Our result takes advantage of the symmetric role of C and D.

Corollary 10. *Let* $g\colon X \mapsto Y$ *be a mapping of class* $\mathcal{C}^1$ *at* $\bar{x}$ *and let* C *and* D *be closed sets in* X *and* Y, *respectively. Consider the system*

$$x \in C, \qquad g(x) \in D$$

to which we associate the parametric solution set given by the multivalued mapping

$$S_5(y) = \{x \in C : g(x) + y \in D\}.$$

Let $\bar{x} \in C \cap g^{-1}(D)$. *Suppose that either*

(i) $Dg(\bar{x})$ *is surjective and* C *is compactly epi-Lipschitzian at* $\bar{x}$, *or*
(ii) D *is compactly epi-Lipschitzian at* $g(\bar{x})$.

Then $(\beta) \Longrightarrow (\alpha)$, *where*

(α) *there exist* $a > 0$ *and* $r > 0$ *such that*

$$d(x, S_5(y)) \leq ad(g(x) + y, D), \qquad \forall x \in C \cap B(\bar{x}, r), \quad \forall y \in B(0, r);$$

and

(β) *there is no* $y^* \in Y^*$, $y^* \neq 0$, *satsfying*

$$-y^* \circ Dg(\bar{x}) \in k_g \partial_A d(C, \bar{x}), \qquad y^* \in k_g \partial_A d(D, g(\bar{x})),$$

where k_g *is a Lipschitz constant of* g *at* $\bar{x}$.

5. Applications

The main intention of this section is devoted to applications of our results to the notion of weak sharp minima, necessary optimality conditions and sensitivity analysis as well as to local controllability of optimal control problems of unbounded differential inclusions with nonconvex admissible velocity sets.

5.1. *Weak Sharp Minima*

We can apply our results to optimization problems, in particular for studying the notion of weak sharp minima which ensures, for example, the finite convergence of some algorithms.

Consider a function $g\colon X \mapsto \mathbb{R} \cup \{\infty\}$. We say that $S := \arg\min g$ is a set of weak sharp minima for g with modulus $b > 0$ if

$$g(x) \geq g(u) + bd(x, S), \qquad \forall x \in X, \quad \forall u \in S.$$

As we can see, this is equivalent to the error bound

$$d(x, S) \leq \frac{1}{b} \max(0, f(x)), \qquad \forall x \in X,$$

where $f(x) = g(x) - g(u)$ for some $u \in S$. So this inequality is ensured under the assumptions of Theorem 5.

5.2. *Necessary Optimality Conditions*

We consider here optimization problems of the form

$$\min\{f(x, y) : g(x, y) = 0,\ (x, y) \in C \times D\}, \tag{9}$$

where $g\colon X \times Y \mapsto Z$ and $f\colon X \times Y \mapsto \mathbb{R}$ are mappings of class $\mathcal{C}^1$ at $(\bar{x}, \bar{y}) \in C \times D$, with $g(\bar{x}, \bar{y}) = 0$, where C and D are closed sets in X and Y, respectively.

A vector $(\lambda, z^*) \in \mathbb{R}_+ \times Z^*$ is a Fritz–John multiplier of (9) at $(\bar{x}, \bar{y})$ if

$$\|(\lambda, z^*)\| = 1, \tag{10}$$

$$-\lambda \nabla_x f(\bar{x}, \bar{y}) - z^* \circ D_x g(\bar{x}, \bar{y}) \in 2ak_g k_f \partial_{\mathrm{A}} d(C, \bar{x}), \tag{11}$$

$$-\lambda \nabla_y f(\bar{x}, \bar{y}) - z^* \circ D_y g(\bar{x}, \bar{y}) \in 2ak_g k_f \partial_{\mathrm{A}} d(D, \bar{y}). \tag{12}$$

Here k_f and k_g denote the Lipschitz constants of f and g near $(\bar{x}, \bar{y})$ and a is as in assertion (α) of Theorem 8 (with $g(x, y)$ instead of $g(x, y, u)$). These constants are assumed to be at least equal to 1.

For a local solution $(\bar{x}, \bar{y})$ to (9) we denote

- all multipliers (λ, z^*) satisfying (10)–(12) by $\mathrm{FJ}(\bar{x}, \bar{y})$ and
- all multipliers z^* satisfying (11) and (12), with $\lambda = 1$, by $\mathrm{KKT}(\bar{x}, \bar{y})$ (the set of Karush–Kuhn–Tucker multipliers).

The following result is a direct consequence of Theorem 8.

Theorem 11. *Suppose that* $(\bar{x}, \bar{y})$ *is a local solution to the problem* (9). *Then, under the assumptions of Theorem* 8, *with* $g(x, y)$ *instead of* $g(x, y, u)$, $FJ(\bar{x}, \bar{y})$ *is nonempty and weak-star compact in* $\mathbb{R} \times Z^*$. *If in addition assertion* (β) *of Theorem* 8 *holds*, *then* $\mathrm{KKT}(\bar{x}, \bar{y})$ *is nonempty and weak-star compact in* Z^*.

We have to note that if neither (ii) nor (iv) in Theorem 8 is satisfied, then the theorem is wrong. To see this let $X = Y = l^2$ be the Hilbert space of square summable sequences, with (e_k) its canonical orthonormal base and let the operator $A\colon l^2 \to l^2$ be defined by

$$A\left(\sum x_i e_i\right) = \sum 2^{1-i} x_i e_i.$$

Then A is not surjective and $\mathrm{Im}(A)$ is a proper dense subspace of l^2. The adjoint A^* is injective but not surjective. So let $x^* \notin \mathrm{Im}(A^*)$ and set $f = x^*$, $g = A$ and $D = \{0\}$. Then 0 is the only feasible point and it is the optimum for this problem. Moreover, there is no $(\lambda, y^*) \neq (0, 0)$ satisfying $\lambda \nabla f(\bar{x}) + y^* \circ Dg(\bar{x}) = 0$.

5.3. *Sensitivity Analysis*

Suppose that an optimization problem (P) is given in the following abstract form:

$$\min\{f(x, y) : g(x, y) = 0,\ (x, y) \in C \times D\}.$$

It often happens that (P) lends itself naturally to parametric perturbation, so that (P) is embedded in a family of optimization problems (P_u) indexed by a parameter u

$$\min\{f(x, y, u) : g(x, y, u) = 0,\ (x, y) \in C \times D\}$$

where $f\colon X \times Y \times U \mapsto \mathbb{R}$ is a lower semicontinuous function, $g\colon X \times Y \times U \mapsto Z$ is a mapping and C and D are closed sets in X and Y, respectively.

The value of problem (P_u) is denoted $v(u)$, and v is called the value function. For each u in the domain of v we consider the set of minimizers:

$$S(u) := \{(x, y) \in C \times D : g(x, y, u) = 0, f(x, y, u) = v(u)\}.$$

We proceed to examine a few typical properties of v that have a bearing on (P). We begin with the Lipschitzian property of v. For this we introduce a compactness assumption which will assure the stability of the parametrized problems (P_u). A stability assumption **(SA)** holds if there exists a norm-compact set H such that for u near 0, $S(u) \neq \emptyset$ and

$$S(u) \subset H + B(0, \rho(u)),$$

where $\lim_{u \to 0} \rho(u) = 0$.

We have the following properties of the value function v.

Proposition 12. *Suppose that* **(SA)** *holds and that f and g are continuous on $S(0) \times \{0\}$ and $H \times \{0\}$, respectively. Then*:

(a) *The value function v is lower semicontinuous at* 0.
(b) *The following assertions are equivalent*:
  (i) *The multivalued mapping S is upper semicontinuous at* 0; *i.e.*,
  $\forall \varepsilon > 0, \quad \exists \eta > 0, \qquad S(u) \subset S(0) + B(0, \varepsilon), \qquad \forall u \in B(0, \eta).$
  (ii) *The value function v is upper semicontinuous at* 0.

Proof. (a) Suppose the contrary, then there exist $\varepsilon > 0$ and a sequence (u_n) converging to 0 such that for n large enough,

$$v(0) > v(u_n) + \varepsilon.$$

By **(SA)** there exists $(x_n, y_n) \in S(u_n)$, which we assume converges to some $(\bar{x}, \bar{y})$. Now from the continuity of f and g we deduce

$$v(0) \geq f(\bar{x}, \bar{y}, 0) + \varepsilon, \qquad (\bar{x}, \bar{y}) \in C \times D, \qquad g(\bar{x}, \bar{y}, 0) = 0$$

and hence

$$v(0) \geq v(0) + \varepsilon,$$

which leads to a contradiction. So v is lower semicontinuous at 0.

(b) Suppose that (i) holds. Let (u_n) be any sequence converging to 0 and for which $\lim_{n\to+\infty} v(u_n)$ exists. We will show that $\lim_{n\to+\infty} v(u_n) = v(0)$. By **(SA)** there exists $(x_n, y_n) \in S(u_n)$, which we assume converges to $(\bar{x}, \bar{y})$ and by (i), $(\bar{x}, \bar{y}) \in S(0)$. Thus

$$v(u_n) = f(x_n, y_n, u_n), \qquad (x_n, y_n) \in C \times D, \qquad g(x_n, y_n, u_n) = 0$$

and by the continuity of f and g we get

$$\lim_{n\to+\infty} v(u_n) = f(\bar{x}, \bar{y}, 0), \qquad (\bar{x}, \bar{y}) \in C \times D, \qquad g(\bar{x}, \bar{y}, 0) = 0.$$

As $(\bar{x}, \bar{y}) \in S(0)$, we obtain $\lim_{n\to+\infty} v(u_n) = v(0)$. Now it suffices to use these arguments to prove that

$$\limsup_{u\to 0} v(u) = v(0).$$

Conversely, suppose that v is upper semicontinuous at 0 and that S is not upper semicontinuous at 0. Then there are $\varepsilon > 0$ and sequences (u_n) and $((x_n, y_n))$ such that

$$(x_n, y_n) \in S(u_n) \quad \text{and} \quad (x_n, y_n) \notin S(0) + B(0, \varepsilon).$$

We may assume, by **(SA)**, that $(x_n, y_n) \to (\bar{x}, \bar{y})$. Since

$$v(u_n) = f(x_n, y_n, u_n), \qquad (x_n, y_n) \in C \times D, \qquad g(x_n, y_n, u_n) = 0,$$

then by the continuity of f and g and the upper semicontinuity of v at 0 we obtain

$$v(0) \geq \limsup_{n\to+\infty} v(u_n) = f(\bar{x}, \bar{y}, 0), \qquad (\bar{x}, \bar{y}) \in C \times D, \qquad g(\bar{x}, \bar{y}, 0) = 0,$$

which is equivalent to saying that $(\bar{x}, \bar{y}) \in S(0)$. Thus, for n large enough, $(x_n, y_n) \in S(0) + B(0, \varepsilon)$ and this contradiction completes the proof. □

Theorem 13. *Suppose that*:

1. *For each sequence* (u_n) *converging to* 0 *we have*

$$\emptyset \neq \limsup_{n\to+\infty} S(u_n) \subset S(0).$$

2. *For each* $(\bar{x}, \bar{y}) \in S(0)$ *we have*:
   (i) f, g *are locally Lipschitzian near* $(\bar{x}, \bar{y}, 0)$ *with Lipschitz constant* $k(\bar{x}, \bar{y})$.

(ii) *g is of class $\mathcal{C}^1$ at $(\bar{x}, \bar{y}, 0)$ in (x, y) with respect to u with surjective partial derivative $D_x g(\bar{x}, \bar{y}, 0)$.*
(iii) *f is of class $\mathcal{C}^1$ at $(\bar{x}, \bar{y}, 0)$ in (x, y) with respect to u.*
(iv) *C is compactly epi-Lipschitzian at $\bar{x}$.*
(v) *Assertion (β) of Theorem 8 holds.*

Then v is locally Lipschitzian near 0.

Proof. We proceed to show that v is locally Lipschitzian around 0. So suppose the contrary, then there are sequences $u_n \to 0$ and $u'_n \to 0$ such that, for n large enough,

$$|v(u_n) - v(u'_n)| > nd(u_n, u'_n).$$

We may assume that the set $I = \{n : v(u_n) - v(u'_n) > nd(u_n, u'_n)\}$ is infinite (because (u_n) and (u'_n) play a symmetric role). For all $n \in I$ there exists, by assumption 1, $((x'_n, y'_n))_{n \in J \subset I}$ which converges to $(\bar{x}, \bar{y}) \in S(0)$ and $(x'_n, y'_n) \in S(u'_n)$, for all $n \in J$. Now, by Theorem 8, for $n \in J$ large enough,

$$d((x'_n, y'_n), S_3(u_n)) \leq a\|g(x'_n, y'_n, u_n)\|$$

and hence there exists $(x_n, y_n) \in S_3(u_n)$ such that

$$\|(x'_n, y'_n) - (x_n, y_n)\| \leq a\|g(x'_n, y'_n, u_n)\|$$

and since g is locally Lipschitzian near 0 uniformly in (x'_n, y'_n), with constant $k_g = k_g(\bar{x}, \bar{y})$,

$$\|(x'_n, y'_n) - (x_n, y_n)\| \leq a\|g(x'_n, y'_n, u_n) - g(x'_n, y'_n, u'_n)\| \leq ak(\bar{x}, \bar{y})d(u_n, u'_n).$$

Then, for all $n \in I$ sufficiently large,

$$nd(u_n, u'_n) < f(x_n, y_n, u_n) - f(x'_n, y'_n, u'_n) \leq k(\bar{x}, \bar{y})(1 + ak(\bar{x}, \bar{y}))d(u_n, u'_n)$$

and this contradiction completes the proof. □

Corollary 14. *The result of Theorem 13 remains valid if we replace assumption 1 by the following assumption*:

1′. **(SA)** *holds and S is upper semicontinuous at 0.*

Let $\mathrm{KKT}(\bar{x}, \bar{y})$ denote the set of Karush–Kuhn–Tucker multipliers of (P_0) at $(\bar{x}, \bar{y})$, that is, the set of $z^* \in Z^*$ satisfying

$$-\nabla_x f(\bar{x}, \bar{y}, 0) - z^* \circ D_x g(\bar{x}, \bar{y}, 0) \in 6(1 + ak_g)(k_v + k_f)\partial_{\mathrm{A}} d(C, \bar{x}),$$
$$-\nabla_y f(\bar{x}, \bar{y}, 0) - z^* \circ D_y g(\bar{x}, \bar{y}, 0) \in 6(1 + ak_g)(k_v + k_f)\partial_{\mathrm{A}} d(D, \bar{y}).$$

Here k_v, k_f and k_g denote Lipschitz constants of v near 0 and f and g near $(\bar{x}, \bar{y}, 0)$ and a is as in assertion (α) of Theorem 8. These constants are assumed to be at least equal to 1.

Then we have the following estimate of the subdifferential of v.

Theorem 15. *Suppose in addition to the assumptions of Theorem* 13 *that f and g are of class $\mathcal{C}^1$ at $(\bar{x}, \bar{y}, 0)$ for each $(\bar{x}, \bar{y}) \in S(0)$ and that the perturbation set U is a Banach space. Then*

$$\partial_{\mathrm{A}} v(0) \subset \bigcup_{(\bar{x},\bar{y})\in S(0)} \{\nabla_u f(\bar{x}, \bar{y}, 0) + z^* \circ D_u g(\bar{x}, \bar{y}, 0) : z^* \in \mathrm{KKT}(\bar{x}, \bar{y})\}.$$

Proof. The proof is similar to that in [11]. Let k_v be a Lipschitz constant of v around 0 (which is possible since, by Theorem 13, v is locally Lipschitzian near 0). Let $u^* \in \partial_{\mathrm{A}} v(0)$. Then, by Proposition 1, we have, for all $L \in \mathcal{F}(U)$, that there exist nets $u_i \to 0$, $\varepsilon_i \to 0^+$, $u_i^* \to u^*$, with $\|u_i^*\| \leq (k_v + 1)(1 + \varepsilon_i)$, and $r_i \to 0^+$ such that, for all $u \in B(u_i, r_i)$,

$$v(u) - v(u_i) - \langle u_i^*, u - u_i \rangle + \varepsilon_i \|u - u_i\| + 2(k_v + \varepsilon_i) d(u, u_i + L) \geq 0.$$

From assumption 1 in Theorem 13 there exist $(\bar{x}, \bar{y}) \in S(0)$ and $(x_i, y_i) \in S(u_i)$, with $(x_i, y_i) \to (\bar{x}, \bar{y})$, such that for all $(x, y, u) \in C \times D \times B(u_i, r_i)$, $g(x, y, u) = 0$, we have

$$\begin{aligned} &f(x, y, u) - f(x_i, y_i, u_i) - \langle u_i^*, u - u_i \rangle + \varepsilon_i \|u - u_i\| \\ &\quad + 2(k_v + \varepsilon_i) d(u, u_i + L) \geq 0. \end{aligned}$$

Using Theorem 8 we obtain

$$\begin{aligned} &3a(k_f + k_v)\|g(x, y, u)\| + f(x, y, u) - f(x_i, y_i, u_i) - \langle u_i^*, u - u_i \rangle \\ &\quad + \varepsilon_i \|u - u_i\| + (k_v + \varepsilon_i) d(u, u_i + L) \geq 0 \end{aligned}$$

for all $(x, y, u) \in C \cap B(x_i, r_i) \times D \cap B(y_i, r_i) \times B(u_i, r_i)$. Thus the function

$$\begin{aligned} &(x, y, u) \mapsto 6(1 + ak_g)(k_f + k_v)[d(x, C) + d(y, D)] + 2a(k_f + k_v)\|g(x, y, u)\| \\ &\quad + f(x, y, u) - f(x_i, y_i, u_i) - \langle u_i^*, u - u_i \rangle + \varepsilon_i \|u - u_i\| + 3k_v d(u, u_i + L) \end{aligned}$$

attains its local minimum at (x_i, y_i, u_i). We conclude by using subdifferential calculus and by passing to the limit. □

In the case where f and g are not depending on the perturbation u and $g = g_1 - g_2$, where $g_1\colon X \mapsto Z$ and $g_2\colon Y \mapsto Z$, then we get the following result which is a direct consequence of the previous one.

Corollary 16. *Under the assumptions of Theorem* 15 *we have*

(i) *for all $(\bar{x}, \bar{y}) \in S(0)$, $\partial_{\mathrm{C}} v(0) \cap \mathrm{KKT}(\bar{x}, \bar{y}) \neq \emptyset$ and*

(ii)

$$\partial_A v(0) \subset \bigcup_{(\bar{x},\bar{y})\in S(0)} \mathrm{KKT}(\bar{x},\bar{y}).$$

Here ∂_C *denotes Clarke's subdifferential.*

Proof. It suffices to prove the first part. Let $(\bar{x}, \bar{y}) \in S(0)$. Then

$$f(\bar{x},\bar{y}) - v(0) = 0 \le f(x,y) - v(u)$$

for all (x, y, u) near $(\bar{x}, \bar{y}, 0)$, with $(x, y) \in S_3(u)$. By Theorem 8 there exists constant $a > 0$ such that

$$d((x,y), S(u)) \le \|g_1(x) + u - g_2(y)\|$$

for all (x, y, u) near $(\bar{x}, \bar{y}, 0)$, with $(x, y) \in C \times D$. So that $(\bar{x}, \bar{y}, 0)$ is a local solution of the function

$$(x,y,u) \mapsto f(x,y) - v(u) + a(k_f + k_v)\|g_1(x) + u - g_2(y)\|$$
$$+ 2a(k_f + k_v)[d(x,C) + d(y,D)].$$

So the conclusion follows by using the subdifferential calculus. □

5.4. *Local Controllability*

We consider here systems of the form

$$\dot{x}(t) \in F(t, x(t)) \qquad \text{a.e.} \quad t \in [a,b], \qquad (x(a), x(b)) \in S, \tag{13}$$

where $F\colon [a,b] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a multivalued mapping which is measurable in the first variable $t \in [a,b]$ and $S \subset \mathbb{R}^n \times \mathbb{R}^n$ is a nonempty closed set. The domain over which the study of system (13) occurs is typically one of the functions $W^{1,p}([a,b],\mathbb{R}^n)$ (abbreviated $W^{1,p}$) consisting of all absolutely continuous functions $x\colon [a,b] \mapsto \mathbb{R}^n$ for which $|\dot{x}|$ is in the functional space $L^p([a,b],\mathbb{R}^n)$ (abbreviated L^p) ($\dot{x}$ denotes the derivative (almost everywhere) of x). The space $W^{1,p}$ is endowed with the norm

$$\|x\| = |x(a)| + \|\dot{x}\|_{L^p},$$

where $|\cdot|$ denotes the euclidean norm of $\mathbb{R}^n$. Here we assume that $p \ge 1$.

Consider the multivalued mapping $G\colon \mathbb{R}^n \mapsto W^{1,p}$ defined by

$$G(y) = \{x \in W^{1,p}\colon \dot{x}(t) \in F(t, x(t)) \text{ a.e., } (x(a), x(b) + y) \in S\}. \tag{14}$$

The distance function on $W^{1,p}$ or $\mathbb{R}^n \times \mathbb{R}^n$ will be denoted by $d(\cdot,\cdot)$.

Let z be a solution of system (13). This system is said to be *locally controllable* at z if there exist $\alpha > 0$ and $r > 0$ such that

$$G(y) \cap B(z, \alpha|y|) \ne \emptyset, \qquad \forall y \in B(0,r).$$

Let $S = C_a \times C_b$ and C be the solution set of the system

$$x(a) \in C_a, \qquad \dot{x}(t) \in F(t, x(t)) \qquad \text{a.e.} \quad t \in [a, b].$$

Consider the linear continuous mapping $w(x) = x(b)$ and let w^* denote its adjoint mapping.

Theorem 17. *The system is locally controllable at z provided that C is closed (which is the case when the multivalued mapping $x \mapsto F(t, x)$ has a closed graph for almost all t) and*

$$w^*(N_{\mathrm{F}}(C_b, z(b))) \cap -N_{\mathrm{F}}(C, z) = \{0\}. \tag{15}$$

As a consequence of this theorem we obtain the following result.

Corollary 18. *Let $p = 1$. Assume that F is closed-valued and measurably Lipschitzian at z and bounded by a summable function (in L^1) around $z(t)$ a.e. in $[a, b]$. Suppose that if*

$$(\dot{v}(t), v(t)) \in \partial_{\mathrm{C}} d(F(t, \cdot), \cdot)(z(t), \dot{z}(t)) \ a.e., \tag{16}$$

and

$$v(a) \in \partial_{\mathrm{F}} d(z(a), C_a), \qquad v(b) \in \partial_F d(z(b), C_b),$$

then

$$v(b) = 0.$$

Then the conclusion of Theorem 17 *holds.*

Here ∂_{C} refers to Clarke's subdifferential [4].

Proof. It suffices to show that (15) holds and to apply Theorem 17. Indeed, consider (as in [31]) the mappings α: $R^n \times L^1 \to R^n \times R^n$ and β: $R^n \times L^1 \to L^1 \times L^1$ defined by

$$\alpha(x(0), \dot{x}) = (x(a), x(b)), \qquad \beta(x(a), \dot{x}) = (x, \dot{x}).$$

Let $c_b \in N_{\mathrm{F}}(C_b, z(b))$, with $-w^*(c_b) \in N_{\mathrm{F}}(S, z)$. By Proposition 6.3 in [12] there exist $K > 0$, $c_a \in K \partial_{\mathrm{F}} d(z(a), C_a)$ and $(u, v) \in K \partial_{\mathrm{A}} I_L(z, \dot{z})$ such that

$$-\alpha^*(c_a, c_b) = \beta^*(u, v),$$

where $I_L(x, y) = \int_a^b d(y(t), F(t, x(t)))\, dt$. Thus (see [31])

$$c_b = -v(b), \qquad c_a = v(a) \quad \text{and} \quad u(t) = \dot{v}(t), \qquad \text{a.e.}$$

and hence $c_b = 0$ and the proof is complete. □

This corollary has been extended in [13] to the more general class of multivalued mappings, namely the *sub-Lipschitzian* multivalued mappings in the sense of Loewen–Rockafellar [18]. In [13] condition (16) is replaced by the following weaker one:

$$\dot{p}(t) \in \text{co}\, D_F^* F(t, z(t), \dot{z}(t))(-p(t)) \qquad \text{a.e.} \quad t \in [a, b], \tag{17}$$

where $D_F^* F(t, \cdot)$ means the coderivative [24]–[26] of $F(t, \cdot)$ in x at the point $(z(t), \dot{z}(t))$ and "co" stands for convex hull.

Now let C be the solution set of the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \qquad \text{a.e.} \quad t \in [a, b].$$

Consider the linear continuous mapping $w(x) = (x(a), x(b))$ and let w^* denote its adjoint mapping.

Theorem 7 gives us the following result.

Theorem 19. *The system is locally controllable at z provided that C is closed* (*which is the case when the multivalued mapping $x \mapsto F(t, x)$ has a closed graph for almost all t*) *and*

$$w^*(N_{\mathrm{F}}(S, (z(a), z(b)))) \cap -N_{\mathrm{F}}(C, z) = \{0\}.$$

Proof. We consider the following linear continuous mapping $w_1(x, y) = (x(a), x(b) + y)$, then we have that

$$G(y) = \{x \in W^{1,p} : x \in C, w_1(x, y) \in S\}$$

and then $(z, 0)$ is a solution of this system.

It suffices to show that part (β) of Theorem 7 holds for this system. On the contrary, suppose that there exists $y^* \neq 0$ such that

$$y^* \in N_{\mathrm{F}}(S, w_1(z, 0))$$

and

$$0 \in y^* \circ D_x w_1(z, 0) + N_{\mathrm{F}}(C, z).$$

However, we have that

$$D_x w_1(z, 0)(x) = (x(a), x(b)) = w(x)$$

and then

$$y^* \circ D_x w_1(z, 0)(x) = \langle y^*, w(x) \rangle = \langle w^*(y^*), x \rangle$$

for all x, and hence $y^* \circ D_x w_1(z, 0) = w^*(y^*)$, but we have supposed that $y^* \in N_F(S, (z(a), z(b)))$ and then

$$w^*(N_F(S, (z(a), z(b)))) \cap -N_F(C, z) \neq \{0\},$$

which leads to a contradiction with the hypothesis of the theorem. So part (β) of Theorem 7 holds and then there exist $a > 0$ and $r > 0$ such that

$$d(x, G(y)) \leq ad(w_1(x, y), S)$$

for all $x \in C \cap B(z, r)$ and all $y \in B(0, r)$. Therefore

$$d(x, G(y)) \leq ad((x(a), x(b) + y), S)$$

and evaluating in z we obtain

$$d(z, G(y)) \leq ad((z(a), z(b) + y), S) \leq a\|y\|, \qquad \forall y \in B(0, r),$$

and this completes the proof. □

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