Weierstrass Institute for Applied Analysis and Stochastics
ANNEGRET GLITZKY, JENS A. GRIEPENTROG

Discrete Sobolev–Poincaré inequalities for Voronoi finite volume approximations

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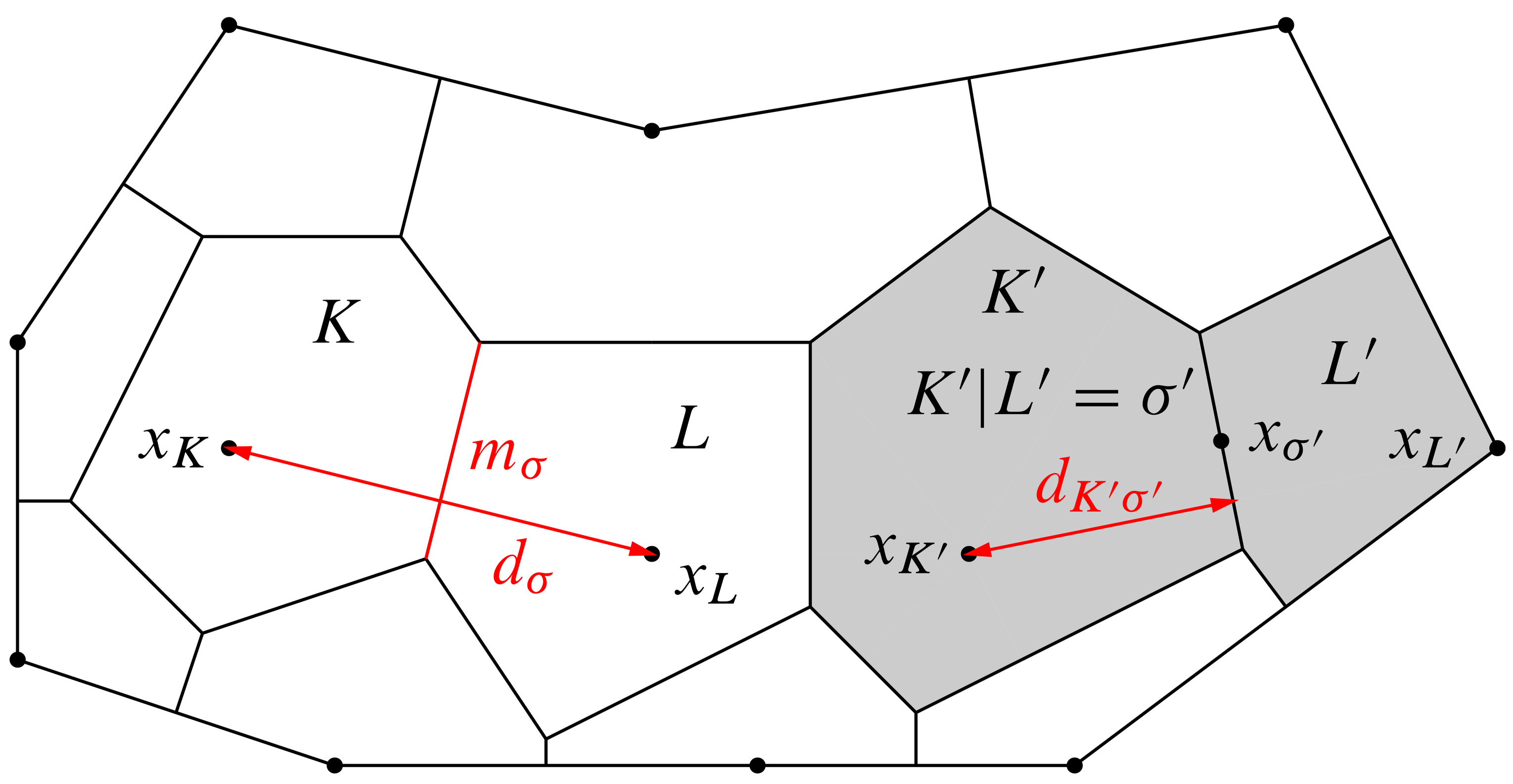
Boundary conforming Delaunay grids for $\Omega \subset \mathbb{R}^n$

– open, polyhedral domain Ω contained in a ball B_R and star shaped with respect to some concentric ball $B_r \subset \Omega$,



Meshes $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$ for Ω

– family $\mathcal{P}$ of grid points x_K in $\overline{\Omega}$,
– family $\mathcal{T}$ of Voronoi control volumes K,
– set $\mathcal{E} = \mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{ext}}$ of interior/exterior Voronoi faces σ,
– subset $\mathcal{E}_K \subset \mathcal{E}$ of faces forming the boundary of $K \in \mathcal{T}$,
– assume that $\mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}} \neq \emptyset$ always implies $x_K \in \partial\Omega$,
– Voronoi face $\sigma = K|L$ between K, $L \in \mathcal{T}$ with surface area m_σ and gravitational center x_σ, Euclidean distance $d_\sigma = |x_K - x_L|$ between x_K and x_L and Euclidean distance $d_{K\sigma}$ between x_K and σ,
– set $X(\mathcal{M})$ of functions $u : \Omega \to \mathbb{R}$ being constant on each $K \in \mathcal{T}$, where u_K is the value of u in the Voronoi box K,
– discrete H^1-seminorm for $u \in X(\mathcal{M})$:

$$|u|^2_{1,\mathcal{M}} = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma}, \quad \text{where } D_\sigma u = |u_K - u_L|.$$

Mesh quality of $\mathcal{M}$

– cut-off function $\rho : \mathbb{R}^n \to [0,1]$ defined as

$$\rho(y) = \begin{cases} \exp\left(r^2/(|y|^2 - r^2)\right) & \text{if } |y| < r, \\ 0 & \text{if } |y| \geq r, \end{cases}$$

– piecewise constant approximation $\rho^{\mathcal{M}} \in X(\mathcal{M})$ given by

$$\rho^{\mathcal{M}}_K(x) = \min_{y \in \overline{K}} \rho(y) \quad \text{for } x \in K,$$

– consider meshes with constants $\rho_0 > 0$, $\kappa_1 > 0$ and $\kappa_2 > 0$ such that $\int_\Omega \rho^{\mathcal{M}}(x)\, dx \geq \rho_0$ and

$$\frac{\operatorname{diam} \sigma}{d_\sigma} \leq \kappa_1 \quad \text{for all } \sigma \in \mathcal{E}_{\mathrm{int}}, \qquad \frac{R_{K,\mathrm{out}}}{R_{K,\mathrm{int}}} \leq \kappa_2 \quad \text{for all } K \in \mathcal{T}$$

– minimal radius $R_{K,\mathrm{out}}$ of balls $B \supset K$ centered at x_K,
– maximal radius $R_{K,\mathrm{int}}$ of balls $B \subset K$ centered at x_K.

Discrete version of Sobolev's integral representation

$$|u_L - m_\Omega(u)| \int_\Omega \rho^{\mathcal{M}}(x)\, dx \leq I_1 + I_2(L) \quad \text{for all } L \in \mathcal{T},$$

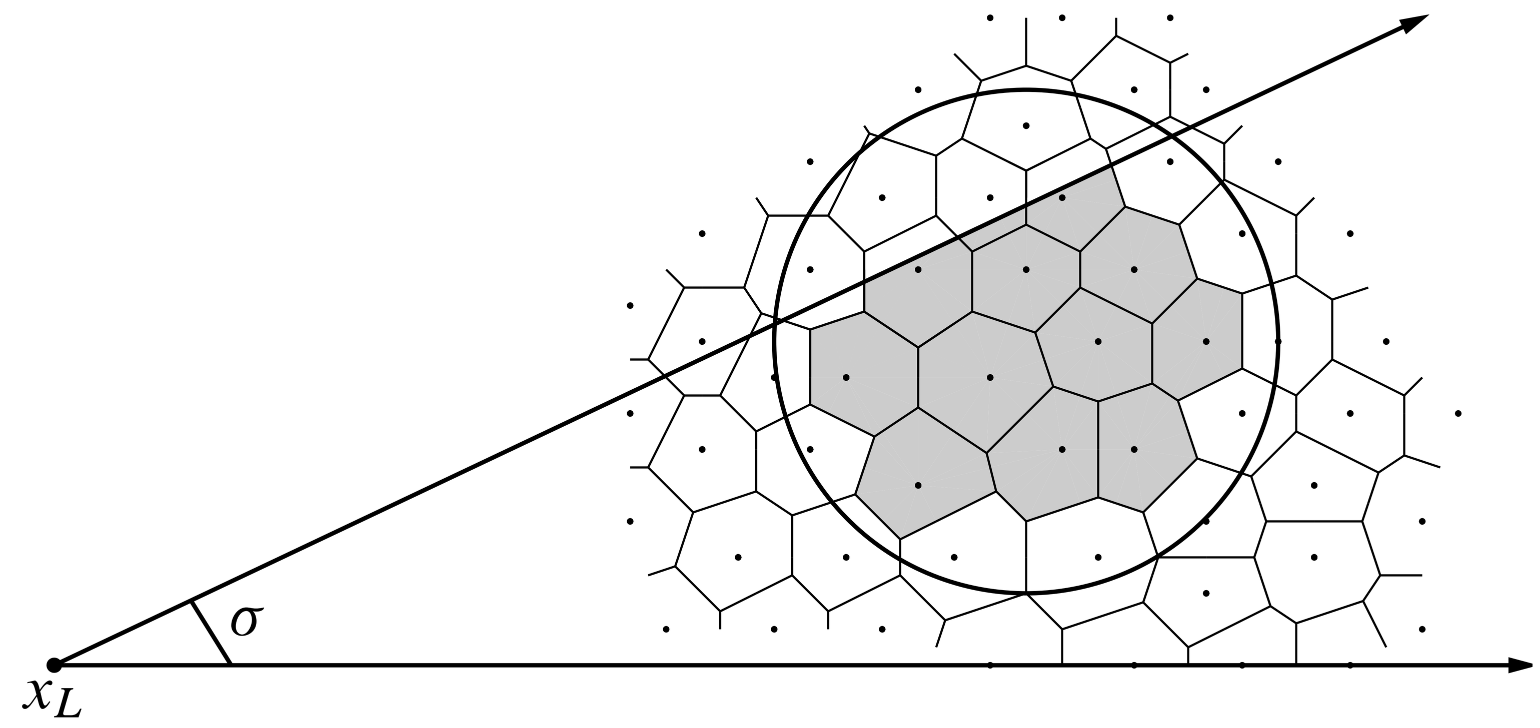
estimated by the sum of integrals

$$I_1 = \int_\Omega |u(x) - m_\Omega(u)|\, \rho^{\mathcal{M}}(x)\, dx,$$

and

$$I_2(L) = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} D_\sigma u \ \operatorname{mes}\left(\{x \in B_r : [x_L, x] \cap \sigma \neq \emptyset\}\right).$$

Parts of Voronoi boxes included in the ball B_r and shaded by the Voronoi surface σ with respect to the viewpoint x_L:



Potential theoretical lemmas

There exists some constant $A = A(R, n, \kappa_1) > 0$ such that for all $L \in \mathcal{T}$ and $\sigma \in \mathcal{E}_{\mathrm{int}}$ the following *solid angle estimate* holds true:

$$\operatorname{mes}\left(\{x \in B_r : [x_L, x] \cap \sigma \neq \emptyset\}\right) \leq \frac{A\, m_\sigma}{|x_\sigma - x_L|^{n-1}}.$$

Let $q \in (2, \infty)$ for $n = 2$, $q \in \left(2, 2n/(n-2)\right)$ for $n \geq 3$, and fix $\beta > 0$ by $2\beta = n/q - n/2 + 1$. There exist constants $B = B(q, n, \kappa_1) > 0$ and $D = D(q, n, \kappa_1, \kappa_2) > 0$ such that the following *weakly singular integral estimates* hold true:

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \frac{m_\sigma\, d_{K\sigma}}{|x_\sigma - x_L|^{n-2\beta}} \leq \int_\Omega \frac{B\, dx}{|x - x_L|^{n-2\beta}} \quad \text{for all } L \in \mathcal{T}$$

and

$$\sum_{L \in \mathcal{T}} \sum_{\tau \in \mathcal{E}_L} \frac{m_\tau\, d_{L\tau}}{|x_L - x_\sigma|^{n-q\beta}} \leq \int_\Omega \frac{D\, dx}{|x - x_\sigma|^{n-q\beta}} \quad \text{for all } \sigma \in \mathcal{E}_{\mathrm{int}}$$

Discrete Sobolev–Poincaré inequality

Let $q \in [1, \infty)$ for $n = 2$ and $q \in \left[1, 2n/(n-2)\right)$ for $n \geq 3$. Then there exists some constant $C > 0$ depending only on n, q, Ω and the mesh constants ρ_0, κ_1, κ_2 such that

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq C\, |u|_{1,\mathcal{M}} \quad \text{for all } u \in X(\mathcal{M}).$$