Cubature and splitting schemes for stochastic differential equations

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Outline

Weak approximation of solutions of SDEs

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i =: \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i, \qquad (1)$$

- $V_0, \ldots, V_d : \mathbb{R}^N \to \mathbb{R}^N$ vector fields
- B_t a d-dimensional Brownian motion, $B_t^0 := t$

Problem

For $f : \mathbb{R}^N \to \mathbb{R}$ sufficiently regular, compute $u(t, x) := E\left[f(X_T)|X_t = x\right]$.

PDE formulation

$\partial_t u + Lu = 0$, where $Lf(x) = V_0 f(x) + \frac{1}{2}\sum_{j=1}^{d} V_i^2 f(x)$, $V_i f(x) := V_i(x) \cdot \nabla f(x)$.

Different approaches

PDE methods: Solve the (linear, second order, parabolic) PDE directly, using finite elements, finite differences,. . . .

Probabilistic methods: Solve the SDE and integrate.

- Discretize SDE to find an approximate solution $\overline{X}_T^{(n)}$.
- Integrate $E\left[f\left(\overline{X}_T^{(n)}\right)\right]$ using (quasi) Monte-Carlo simulation.

Splitting methods: Use structure $L = V_0 + \frac{1}{2}\sum_{j=1}^{d} V_i^2 = \sum_{i=0}^{d} L_i$.

- Solve the PDEs for L_i and combine solutions.
- Probabilistic splitting schemes.

We only consider the probabilistic methods in this talk, with the aim of obtaining higher order methods.

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Euler discretization of SDEs

- SDE: $dX_t = \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i$
- Naive Euler discretization: $\overline{X}_{t_{j+1}}^{(n)} = \overline{X}_{t_j}^{(n)} + V_0(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$
- Scaling property of Brownian increments: $\Delta B_j^i \sim \mathcal{N}(0, \Delta t_j) \approx \sqrt{\Delta t_j}$, $(\Delta B_j^i)^2 \approx \Delta t_j$
- Correct Euler discretization: $\overline{X}_{t_{j+1}} = \overline{X}_{t_j} + V(\overline{X}_{t_j}^{(n)})\Delta t_j + \sum_{i=1}^{d} V_i(\overline{X}_{t_j}^{(n)})\Delta B_j^i$, with $V(x) = V_0(x) + \frac{1}{2}\sum_{i=1}^{d} DV_i(x) \cdot V_i(x)$

Complications compared to discretization of ODEs

- Higher order terms relevant
- "May not look into future."

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- Monte Carlo simulation: $\Delta B^{(l)}$ indep. realizations of ΔB,

$$E\left[f\left(\overline{X}_T^{(n)}\right)\right] \approx \frac{1}{M}\sum_{l=1}^{M} f\left(\overline{X}_T^{(n)}(\Delta B_1^{(l)}, \ldots, \Delta B_n^{(l)})\right)$$

- Integration error stochastic, but of order $1/\sqrt{M}$, independent of the dimension $n \times d$
- Quasi Monte Carlo simulation: take deterministic vectors $\Delta B^{(l)}$ with special “uniformity” properties
- Integration error of order $1/M$ when dimensions not too high.
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Discussion of the probabilistic method

- Order of convergence of Euler scheme: n^{-1} (generically)
- Order of convergence of the (Q)MC simulation: $M^{-1/2}$, M^{-1}
- Integration error dominates.

Goal

Find higher order discretization methods.

- Reduce the dimension $n \times d$ of the integration problem, allowing to rely on quasi Monte Carlo simulation.
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Stochastic Taylor expansion

Ito formula for Stratonovich calculus

$$df(X_t) = V_0 f(X_t)dt + \sum_{i=1}^{d} V_i f(X_t) \circ dB_t^i = \sum_{i=0}^{d} V_i f(X_t) \circ dB_t^i,$$

where $V_i f(x) := V_i(x) \cdot \nabla f(x)$, $X_0 = x$.

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$$f(X_t) = f(x) + \sum_{i=0}^{d} V_i f(x) B_t^i + \sum_{i,j=0}^{d} \int_{0 \le u \le s \le t} V_j V_i f(X_u) \circ dB_u^j \circ dB_s^i,$$

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$$f(X_t) = f(x) + \sum_{i_1=0}^{d} V_{i_1} f(x) B_t^{i_1} + \sum_{i_1,i_2=0}^{d} V_{i_1} V_{i_2} f(x) \int_0^t B_{t_2}^{i_1} \circ dB_{t_2}^{i_2}$$
$$+ \sum_{i_1,i_2,i_3=0}^{d} \int_{0 \le t_1 \le t_2 \le t_3 \le t} V_{i_1} V_{i_2} V_{i_3} f(X_{t_1}) \circ dB_{t_1}^{i_1} \circ dB_{t_2}^{i_2} \circ dB_{t_3}^{i_3},$$

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$$f(X_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1} \cdots V_{i_k} f(x) B_t^{(i_1,\dots,i_k)} + R_m(t,x,f),$$

$\sup_x \sqrt{E\left[R_m^2\right]} = \mathcal{O}(t^{\frac{m+1}{2}})$, $B_t^{(i_1,\dots,i_k)} := \int_{0\le t_1\le\cdots\le t_k\le t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k}$.

Random ODEs

- Let W be a $(d+1)$-dimensional process with paths of bounded variation, define $\widetilde{X}_t = X(W)_t$ by the random ODE

$$\frac{d}{dt}\widetilde{X}_t = \sum_{i=0}^{d} V_i(\widetilde{X}_t)\dot{W}_t^i, \quad \widetilde{X}_0 = x. \tag{2}$$

- Ordinary Taylor expansion:

$$f(\widetilde{X}_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1}\cdots V_{i_k} f(x) W_t^{(i_1,\dots,i_k)} + \widetilde{R}_m(t,x,f)$$

- Remember: Stochastic Taylor expansion

$$f(X_t) = \sum_{k=0}^{m} \sum_{(i_1,\dots,i_k)\in\{0,\dots,d\}^k} V_{i_1}\cdots V_{i_k} f(x) B_t^{(i_1,\dots,i_k)} + R_m(t,x,f)$$

Cubature on Wiener space

Definition

W is a cubature formula on Wiener space of degree m iff
$E\left[W_t^{(i_1,\ldots,i_k)}\right] = E\left[B_t^{(i_1,\ldots,i_k)}\right]$ for $k \leq m$.

- Cubature formulas with finite support exist (Lyons and Victoir)
- Construction of cubature formulas for $m > 5$ interesting open problem
- Fix a grid $0 = t_0 < t_1 < \cdots < t_n = T$, define W by concatenation of independent cubature formulas (of degree m) on the sub-intervals $[t_i, t_{i+1}]$.
- Global error: $E[f(X_T)] - E\left[f(\widetilde{X}_T^{(n)})\right] = O((\max_j \Delta t_j)^{(m-1)/2})$

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Extensions

- Jump diffusions (B. and Teichmann): add jump times to grid, but at most $m/2$ for each initial interval
- May reduce order of cubature method by two for each jump
- Backward SDEs (Crisan and Manolarakis): allows to solve semi-linear parabolic PDEs with cubature methods
- Stochastic PDEs (B. and Teichmann): expectation of an SPDE approximated by an expectation of a random PDE
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Abstract splitting

$$E[f(X_t)|X_0 = x] =: P_t f(x) = \exp\left(t\left(V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2\right)\right) f(x)$$

- General splitting: $V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2 = \sum_{i=0}^{d} U_i$, then approximate $P_t \approx \prod_j e^{t\gamma_j U_{i(j)}}$
- Maximal order of convergence: 2 for positive weights γ

Example

- $Q_t = e^{tU_0} \cdots e^{tU_d}$, $Q_t^* = e^{tU_d} \cdots e^{tU_0}$ (Lie-Trotter splitting or symplectic Euler method)
- $Q_t = \frac{1}{2}(e^{tU_0} \cdots e^{tU_d} + e^{tU_d} \cdots e^{tU_0})$ (symmetrically weighted sequential splitting)

Stochastic splitting

- For a splitting scheme: $e^{\frac{\gamma}{2}V_i^2}f(x) = E[f(Y_\gamma)]$, with

$$dY_t = V_i(Y_t) \circ dB_t^i, \quad Y_0 = x.$$

- $e^{\gamma V_0}f(x) = f(z(\gamma))$, with

$$\dot{z} = V_0(z), \quad z(0) = x.$$

- Advantage: Y_t can be much better approximated than X_t.
- For extrapolation to any order see Oshima, Teichmann and Veluščček.

The Ninomiya-Victoir method

- On a (uniform) grid $0 = t_0 < \cdots < t_n = T$ set $\Delta t_i := t_{i+1} - t_i$, $\Delta B_i^j := B_{t_{i+1}}^j - B_{t_i}^j$, Λ_i Bernoulli-distributed
- Set $\overline{X}_0 = x$ and iteratively

$$\overline{X}_{i+1} := \begin{cases} e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B_i^d V_d} \cdots e^{\Delta B_i^1 V_1} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = 1, \\ e^{\frac{\Delta t_i}{2} V_0} e^{\Delta B_i^1 V_1} \cdots e^{\Delta B_i^d V_d} e^{\frac{\Delta t_i}{2} V_0} \overline{X}_i, & \Lambda_i = -1. \end{cases} \quad (3)$$

- $e^{sV_i} x := z(1)$, where $\dot{z}(t) = sV_i(z(t))$, $z(0) = x$
- Global error: $E[f(X_T)] - E\left[f\left(\overline{X}_n\right)\right] = O((\sup \Delta t_i)^2)$
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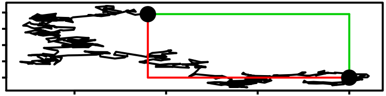

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$$Q_{\Delta t}^{NV} = \frac{1}{2} e^{\frac{\Delta t}{2} L_0} e^{\Delta t L_1} \cdots e^{\Delta t L_d} e^{\frac{\Delta t}{2} L_0} + \frac{1}{2} e^{\frac{\Delta t}{2} L_0} e^{\Delta t L_d} \cdots e^{\Delta t L_1} e^{\frac{\Delta t}{2} L_0},$$

where $L_0 f(x) = V_0 f(x)$, $L_i f(x) = \frac{1}{2} V_i^2 f(x)$,
$Q_{\Delta t}^{NV} \approx P_{\Delta t} := e^{\Delta t L_0 + \Delta t \sum_{i=1}^d L_i}$

ODEs for Ninomiya-Victoir

- Requires $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$
- Numerical solution of ODEs possible, see Ninomiya and Ninomiya.
- Experience suggests that explicit solutions preferable whenever available.
- Question: Which relevant models in mathematical finance allow for explicit formulas of all required terms $\exp(sV_0), \exp(sV_1), \ldots, \exp(sV_d)$?
- Diffusion vector-fields $V_1, \ldots, V_d$ often simple enough, Stratonovich correction causing problems,

$$V_0(x) = V(x) - \frac{1}{2} \sum_{i=1}^{d} DV_i(x) \cdot V_i(x).$$

- Idea: move correction terms back to diffusion part.

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Drift trick

Reformulation

$$dX_t = V_0^{(\gamma)}(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ d(B_t^i + \gamma^i t),$$

where $V_0^{(\gamma)}(x) := V_0(x) - \sum_{i=1}^d V_i(x)\gamma^i$.

- Use Ninomiya-Victoir with V_0 replaced by $V_0^{(\gamma)}$ and ΔB^i replaced by $\Delta B^i + \gamma^i \Delta t$.
- Second order convergence retained.
- Cubature method also obvious.
- Non-standard splitting:
  $L = V_0 + \frac{1}{2}\sum_{i=1}^d V_i^2 = V_0^{(\gamma)} + \sum_{i=1}^d \left(\frac{1}{2}V_i^2 + \gamma^i V_i\right)$

Girsanov transform

- Let $\mathcal{E}_t := \exp\left(\langle\gamma\,, B_t\rangle - \frac{1}{2}\|\gamma\|^2\, t\right)$ and Q be defined by $\frac{dQ}{dP} = \mathcal{E}_T$.
- We have

$$E_P[f(X_T)] = E_Q[f(Y_T)] = E_P[f(Y_T)\mathcal{E}_T],$$

where Y_T solves the SDE with $V_0^{(\gamma)}, V_1, \dots, V_d$.
- But: $\mathrm{Var}[\mathcal{E}_T] = e^{\|\gamma\|^2 T} - 1$.

Generalized SABR model

Model

$$dX_t^1 = a\left(X_t^2\right)^\alpha \left(X_t^1\right)^\beta dB_t^1,$$
$$dX_t^2 = \kappa(\theta - X_t^2)dt + bX_t^2(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2),$$

where $1/2 \leq \alpha, \beta \leq 1$. (SABR: $\alpha = 1, \kappa = 0$.)

$$e^{sV_1}x = \begin{pmatrix} g_1(s,x) \\ x^2 e^{b\rho s} \end{pmatrix}, \quad e^{sV_2}x = \begin{pmatrix} x^1 \\ x^2 e^{b\sqrt{1-\rho^2}s} \end{pmatrix},$$

$$g_1(s,x) = \begin{cases} \left[(1-\beta)\frac{a(x^2)^\alpha}{\alpha b \rho}\left(e^{\alpha b \rho s} - 1\right) + \left(x^1\right)^{1-\beta} \right]_+^{1/(1-\beta)}, & \beta < 1, \\ x^1 \exp\left(\frac{a(x^2)^\alpha}{\alpha b \rho}\left(e^{\alpha b \rho s} - 1\right)\right), & \beta = 1. \end{cases}$$

Generalized SABR model – 2

- No explicit formula for $e^{sV_0}x$, where

$$V_0(x) = \begin{pmatrix} -\frac{1}{2}a^2\beta\left(x^2\right)^{2\alpha}\left(x^1\right)^{2\beta-1} - \frac{1}{2}\alpha ab\rho\left(x^2\right)^{\alpha}\left(x^1\right)^{\beta} \\ \kappa\theta - \left(\kappa + \frac{1}{2}b^2\right)x^2 \end{pmatrix}$$

- Drift trick: choose $\gamma \in \mathbb{R}^d$, set $V_0^{(\gamma)}(x) := V_0(x) - \sum_{i=1}^d \gamma^i V_i(x)$ and consider

$$dX_t = V_0^{(\gamma)}(X_t)dt + \sum_{i=1}^d V_i(X_t) \circ d\left(B_t^i + \gamma^i t\right)$$

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Generalized SABR model – 3

- Choose $\gamma^1 = -\frac{1}{2}\alpha b\rho$, $\gamma^2 = \frac{\alpha b\rho^2 - 2\kappa/b - b}{2\sqrt{1-\rho^2}}$ to obtain

$$V_0^{(\gamma)}(x) = \begin{pmatrix} -\frac{1}{2}a^2\beta\left(x^2\right)^{2\alpha}\left(x^1\right)^{2\beta-1} \\ \kappa\theta \end{pmatrix}$$

- Explicit solution: $e^{sV_0^{(\gamma)}}x = \left(g_0(s,x), \kappa\theta s + x^2\right)$, with

$$g_0(s,x) = \begin{cases} \left[-\theta^2\beta(1-\beta)P(s,x) + \left(x^1\right)^{2(1-\beta)}\right]_+^{1/2(1-\beta)}, & \beta < 1, \\ x^1 \exp\left(-\frac{1}{2}a^2 P(s,x)\right), & \beta = 1, \end{cases}$$

$$P(s,x) = \frac{1}{(2\alpha+1)\kappa\theta}\left(\left(\kappa\theta s + x^2\right)^{2\alpha+1} - \left(x^2\right)^{2\alpha+1}\right)$$

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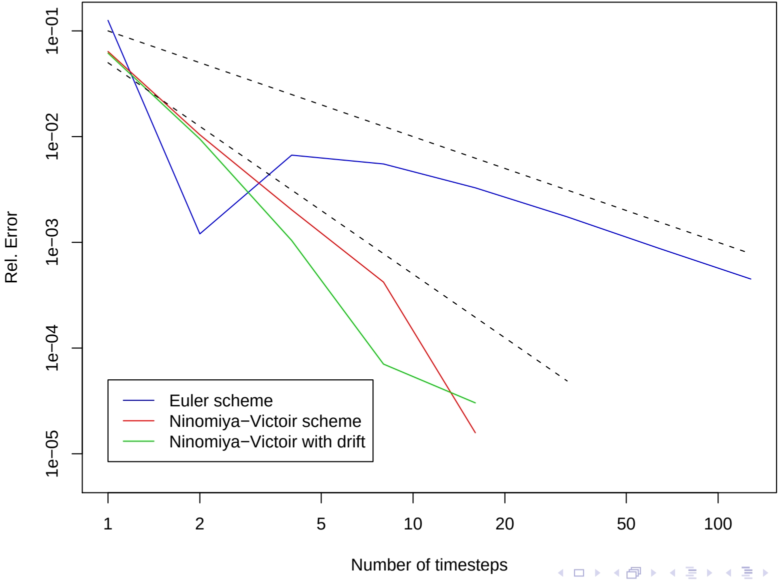
Generalized SABR – Numerical experiment



Generalized SABR – Computational time

Method	n	M	Rel. Error	Time
Euler	32	8192000	0.00174	91.94 sec
Ninomiya-Victoir	4	2048000	0.00204	13.93 sec
NV with drift	4	1024000	0.00104	2.88 sec

Multi-dimensional generalized SABR

Model

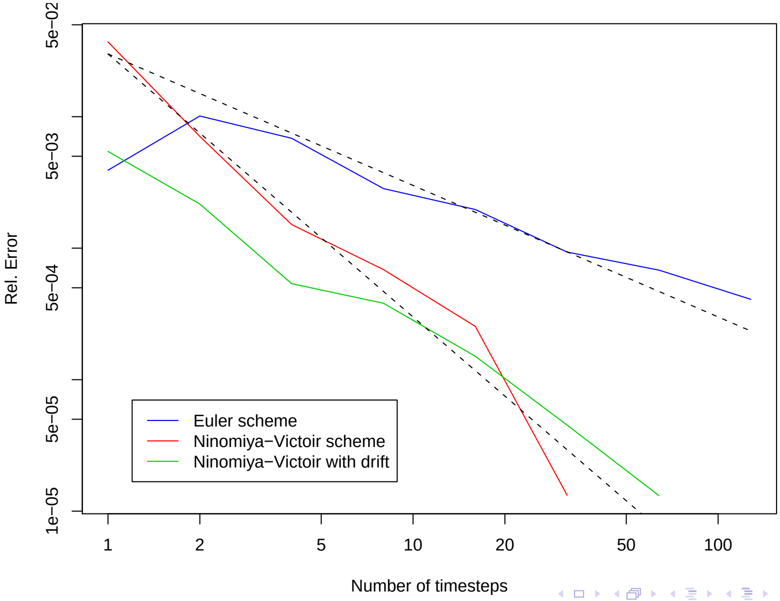
$$dX_i(t) = a_i Y_i(t)^{\alpha_i} X_i(t)^{\beta_i} d\widetilde{B}_t^i$$
$$dY_i(t) = \kappa_i(\theta_i - Y_i(t))dt + b_i Y_i(t) d\widetilde{W}_t^i$$

$\widetilde{B}$ and $\widetilde{W}$ correlated Brownian motions.

- Drift trick allows solving all ODEs explicitly provided that the correlation matrix has full rank.
- Here we use 4 assets, i.e., dimension $N = 8$, $d = 8$.

Method	n	M	Rel. Error	Time
Euler	32	2048000	0.000934	246.65 sec
Ninomiya-Victoir	4	1024000	0.002017	52.33 sec
NV with drift	4	1024000	0.000862	35.31 sec

Multi-dimensional Generalized SABR



Further examples

- More stochastic volatility models qualify:
  - Molina, Han and Fouque
  - Trolle and Schwartz (time-homogeneous version)
- Gatheral's (Bühler's) double mean reverting stochastic volatility model does not fall into this class.

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Construction of splitting and cubature schemes

- Gaussian K schemes: for an m-order approximation Q_t of P_t, find a random variable $Z_{t,x,f}$ s.t. $E[Z] = Q_t f(x)$.
- Approximate $\exp\left(t\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right)\right) \approx E[e^Y]$ for Y taking values in the (step-m nilpotent) free Lie algebra generated by $v_0, \ldots, v_d$.
- Construction of Y comparable to construction of classical cubature formulas on $\mathbb{R}^d$.
- Link to cubature on Wiener space: $\exp\left(t\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right)\right)$ can be interpreted as expectation of the random variable $(B_t^{(i_1,\ldots,i_k)})_{k\leq m}$.

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Iterating the scheme

- Given $\|P_t - Q_t\| \leq t^{\ell+1}$ obtained by cubature, splitting, ...
- Time grid: $0 = t_0 < t_1 < \cdots < t_N = T$, $Q_T^{(N)} := Q_{\Delta t_N} \cdots Q_{\Delta t_1}$.

$$\left\| P_T f - Q_T^{(N)} f \right\|_\infty \leq \sum_{k=1}^{N} \left\| Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}} P_{t_k} f - Q_{\Delta t_N} \cdots Q_{\Delta t_k} P_{t_{k-1}} f \right\|_\infty$$

$$\leq \sum_{k=1}^{N} \left\| Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}} \right\| \left\| (P_{\Delta t_k} - Q_{\Delta t_k}) P_{t_{k-1}} f \right\|_\infty$$

$$\leq \mathrm{const} \sum_{k=1}^{N} \Delta t_k^{l+1} \leq \mathrm{const} \left(\max_k \Delta t_k \right)^{\ell}.$$

- Relax regularity assumptions under Hörmander type conditions.

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$$\begin{aligned}\left\|P_T f - Q_T^{(N)} f\right\|_\infty &\leq \sum_{k=1}^{N} \left\|Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}} P_{t_k} f - Q_{\Delta t_N} \cdots Q_{\Delta t_k} P_{t_{k-1}} f\right\|_\infty \\ &\leq \sum_{k=1}^{N} \left\|Q_{\Delta t_N} \cdots Q_{\Delta t_{k+1}}\right\| \left\|(P_{\Delta t_k} - Q_{\Delta t_k}) P_{t_{k-1}} f\right\|_\infty \\ &\leq \mathrm{const} \sum_{k=1}^{N} \Delta t_k^{l+1} \leq \mathrm{const} \left(\max_k \Delta t_k\right)^{\ell}.\end{aligned}$$

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