Adaptive weak approximation of reflected diffusions

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The Skorohod problem

Given a domain $D \subset \mathbb{R}^d$, consider an SDE reflected at the boundary $\partial D$.

\[
\begin{aligned}
    dX_t^x &= V(X_t^x)dt + \sum_{i=1}^{d} V_i(X_t^x)dB_t^i + n(X_t^x)dZ_t^x, \\
    X_0^x &= x \in D, \quad Z_0^x = 0.
\end{aligned}
\]

- $B_t = (B_t^1, \ldots, B_t^d)$ $d$-dimensional Brownian motion
- $V, V_1, \ldots, V_d : \overline{D} \to \mathbb{R}^d$ vector fields
- $X_t^x \in \overline{D}$, $n(X_t^x)$ inward pointing normal vector at $X_t^x \in \partial D$
- $Z_t^x$ increasing process with $dZ_t^x = 1_{\partial D}(X_t^x)dZ_t^x$ ("local time")
Kolmogorov equation

\[
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) &= -Lu(t, x), \quad (t, x) \in [0, T] \times D, \\
u(T, x) &= f(x), \quad x \in D, \\
\frac{\partial}{\partial n} u(t, x) &= h(x), \quad x \in \partial D.
\end{aligned}
\]  

(2)

\(L\) is the infinitesimal generator of the SDE.

Proposition

Under suitable regularity condition, the solution of (2) has the stochastic representation

\[
u(t, x) = E \left[ f(X_T) - \int_t^T h(X_s) dZ_s \bigg| X_t = x \right].
\]  

(3)
Kolmogorov equation

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\frac{\partial}{\partial t} u(t, x) &= -Lu(t, x), \quad (t, x) \in [0, T] \times D, \\
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**Proposition**

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u(t, x) = E \left[ f(X_T) - \int_t^T h(X_s) dZ_s \right | X_t = x].
\]
The Euler scheme for reflected diffusions

Given a partition $0 = t_0 < t_1 < \ldots < t_N$, let $\Delta t_i = t_{i+1} - t_i$, $\Delta B_i = B_{t_{i+1}} - B_{t_i}$.

**Algorithm**

1. Set $\bar{X}_0 = x$, $\bar{Z}_0 = 0$, $i = 0$.

2. $\hat{X}_{i+1} = \bar{X}_i + V(\bar{X}_i)\Delta t_i + \sum_{j=1}^{d} V_j(\bar{X}_i)\Delta B_j^i$

3. If $\hat{X}_{i+1} \in \overline{D}$ set
   
   $\bar{X}_{i+1} = \hat{X}_{i+1}$, $\bar{Z}_{i+1} = \bar{Z}_i$,

   else set
   
   $\bar{X}_{i+1} = \Pi(\hat{X}_{i+1})$, $\bar{Z}_{i+1}$

4. If $i < N - 1$ increase $i$ by 1 and return to (1), else stop.
The Euler scheme for reflected diffusions

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1. Set $\overline{X}_0 = x$, $\overline{Z}_0 = 0$, $i = 0$.

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   \overline{X}_{i+1} = \hat{X}_{i+1}, \quad \overline{Z}_{i+1} = \overline{Z}_i,
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   \overline{X}_{i+1} = \Pi(\hat{X}_{i+1}), \quad \overline{Z}_{i+1}
   \]

4. If $i < N - 1$ increase $i$ by 1 and return to (1), else stop.
The Euler scheme for reflected diffusions – 2

- Order of convergence: $u(0, x) - \bar{u}(0, x) = \mathcal{O}(\sqrt{\Delta t_{\text{max}}})$
- Algorithm by Gobet [2001] based on approximation of domain by half-spaces (order 1 for particular direction of reflection)
- Algorithm by Bossy, Gobet, Talay [2004] with order 1 for $h \equiv 0$
Turning adaptive

- Goal: given a tolerance level TOL, minimize the computational work such that the computational error remains smaller than TOL.
- Concentrate *only* on the error of the time discretization of the SDE, not the integration error.
- Adaptive algorithms for SDEs (without reflection) given by Szepessy, Tempone and Zouraris [2001]; we concentrate on the error introduced by the reflection.
The minimization problem

- The error representation has leading order term
  \[ E \left[ \sum_{i=0}^{N-1} \Delta Z_i^2 \left\| \frac{\partial^2}{\partial n^2} u(t_{i+1}, X_{i+1}) \right\| \right]. \quad (4) \]

- Corresponding work given by \( \tilde{N} \), the number of hits at the boundary. Use
  \[ E[\tilde{N}] \approx E \left[ \sum_{i=0}^{N-1} \frac{\Delta Z_i}{\sqrt{\Delta t_i}} \right]. \quad (5) \]

- Problem: minimize (5) given that (4) is bounded by TOL among the set of partitions of \([0, T]\).

- Solve by Lagrangian relaxation.
The refinement algorithm

Given a partition $t_0 = 0 < t_1 < \cdots < t_N = T$ and corresponding values for $\Delta B_i$, $\Delta \bar{Z}_i$, $\bar{X}_i$, $i = 0, \ldots, N - 1$.

**Algorithm**

For $i = 0, \ldots, N$ do:

- Compute

$$\ell_i = \frac{1}{2\sqrt{2\pi}} E(\tilde{N}) \left| \frac{\partial^2}{\partial n^2} \nu(t_i, \bar{X}_i) \right| \exp \left( - \frac{d(\bar{X}_i)^2}{2\Delta t_i} \right) \Delta t_i.$$

- If $\ell_i > $ TOL refine the mesh by inserting $t = \frac{t_{i+1} + t_i}{2}$ in the partition. Sample the corresponding value of the Brownian motion $B_t$ using the Brownian bridge.
A mixed boundary condition

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) &= -\frac{1}{2} \Delta u(t, x), \quad (t, x) \in [0, 2] \times D, \\
\frac{\partial}{\partial n} u(t, x) &= x_1, \quad (t, x) \in [0, 2] \times D_N, \\
\end{align*}
\]

\[
\begin{align*}
u(2, x) &= 10 \exp\left(-\sqrt{(10 - x_1)^2 + x_2^2}\right), \quad x \in D, \\
u(t, x) &= 10 \exp\left(-\sqrt{(10 - x_1)^2 + x_2^2}\right), \quad x \in \times(\partial D \setminus D_N).
\end{align*}
\]
Implementation

- Combine the adaptive Euler scheme for reflected diffusions with the adaptive Euler scheme for stopped diffusions by Dzougoutov et al. [2005].

- Approximate

\[ \left| \frac{\partial^2}{\partial n^2} v(t_i, \overline{X}_i) \right| \approx \frac{1}{\| \overline{X}_i - x_{\text{sing}} \|^\beta + \text{TOL}^\alpha}, \]

where \( x_{\text{sing}} = (10, 0) \).

- Used \( \alpha = 2 \) and \( \beta = 0, 1/2 \).
An adaptive algorithm for reflected diffusions

Numerical example

Problem description
Implementation
Results

Result

Average number of timesteps

Error

Uniform
Adaptive (b=0)
Adaptive (b=0.5)
Reference lines

Christian Bayer
Adaptivity for reflected diffusions
References


