Calculation of the Greeks using Malliavin Calculus

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09/25/2006 – SCS Seminar
09/26/2006 – Mathematics Seminar

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Part I

Greeks and Malliavin Calculus in Finance

The Model

Let $B_t = (B_t^1, \dots, B_t^d)$, $t \in [0, T]$, be a d-dimensional Brownian motion on the Wiener space $(\Omega, \mathcal{F}, P)$. $\mathcal{F}_t$ denotes the filtration generated by B and we assume that $\mathcal{F} = \mathcal{F}_T$. We model a financial market, in which $n + 1$ assets are traded, $n \leq d$.

- S_t^0, $t \in [0, T]$, is the "bank account" earning a risk free interest rate $r > 0$ (continuous compounding).
- $S_t = (S_t^1, \dots, S_t^n)$ gives the risky assets ("stocks").

For bounded, measurable functions $a : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ we set

$$dS_t^0 = rS_t^0 dt$$

$$dS_t^i = a^i(t, S_t)S_t^i dt + \sum_{j=1}^d \sigma^{ij}(t, S_t)S_t^i dB_t^j, \;\; i = 1, \dots, n.$$

Strategies

A strategy describes the amount of money invested in each available asset at any given time $t \in [0, T]$.

Definition

A *(self-financing) strategy* is a predictable, $\mathbb{R}^n$-valued process π such that "all of the following integrals are well-defined".

Remark

- *The strategy is self-financing, because we neither allow consumption nor external money entering the financial market. Thus, strategies are determined by n coordinates.*
- *All positions are allowed to be negative (short selling).*

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The Wealth Process of a Portfolio

The wealth process $X^{x,\pi}$ with initial capital x associated to the strategy π is defined by $X_0^{x,\pi} = x$ and its dynamics

$$dX_t^{x,\pi} = \sum_{i=1}^{n} \frac{\pi_t^i}{S_t^i} dS_t^i + \frac{X_t^{x,\pi} - \sum_{i=1}^{n} \pi_t^i}{S_t^0} dS_t^0.$$

Definition

A strategy is *admissible* if the corresponding wealth process is bounded from below by some fixed real number.

The restriction to admissible strategy is economically sensible and disallows "doubling strategies".

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No Arbitrage

Definition

An *arbitrage opportunity* is an admissible strategy π such that $P\left(X_T^{0,\pi} \geq 0\right) = 1$ and $P\left(X_T^{0,\pi} > 0\right) > 0$.

Definition

Let Q be a probability measure on $(\Omega, \mathcal{F})$ equivalent to P. Q is called *equivalent (local) martingale measure* if the discounted price process $\widetilde{S}_t = e^{-rt} S_t$, $t \in [0, T]$, is a (local) martingale under Q.

Remark

The Fundamental Theorem of Asset Pricing roughly says that the existence of (local) martingale measures is equivalent to the non-existence of arbitrage opportunities.

No Arbitrage – 2

Proposition

Assume there exists a predictable, $\mathbb{R}^d$-valued process θ such that

(i) $a^i(t, S_t) - r = \sum_{j=1}^d \sigma^{ij}(t, S_t)\theta^j(t), \quad i = 1, \dots, n,$

(ii) $\int_0^T \|\theta(t)\|^2 \, dt < \infty \quad a.\ s.,$

(iii) $E\left(\exp\left(-\int_0^T \langle\theta(t), dB_t\rangle - \frac{1}{2}\int_0^T \|\theta(t)\|^2 \, dt\right)\right) = 1.$

Then our model is free of arbitrage and the probability measure Q with density $Z_T = \exp\left(-\int_0^T \langle\theta(t), dB_t\rangle - \frac{1}{2}\int_0^T \|\theta(t)\|^2 \, dt\right)$ is a martingale measure.

Consequently, the dynamics of the model under Q are given by

$$dS_t^i = rS_t^i dt + \sum_{j=1}^d \sigma^{ij}(t, S_t) S_t^i dW_t^j, \quad i = 1, \dots, n,$$

where W denotes a Brownian motion under Q.

Complete and Incomplete Markets

Definition

A *contingent claim* is an $\mathcal{F}_T$-measurable, Q-absolutely integrable random variable. A contingent claim Y is *attainable* or *replicable* if there is an admissible, self-financing portfolio π and a number x such that $X_T^{x,\pi} = Y$ a. s. π is called *replicating portfolio* for Y.

Definition

A financial market is *complete*, if every contingent claim is replicable. Otherwise, it is called *incomplete*.

Proposition

Our market is complete if and only if $n = d$ and $\sigma(t, S_t(\omega))$ is invertible for $dt \otimes P$ - a. e. $(t, \omega) \in [0, T] \times \Omega$.

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Complete and Incomplete Markets – 2

Remark

Under realistic conditions, completeness is a very strong property. Many experts agree that realistic models are not complete.

Assumptions

From now on, we assume that our model is arbitrage-free and complete.

Pricing Contingent Claims

Definition

$x \in \mathbb{R}$ is an arbitrage-free price of a contingent claim Y if there is an admissible strategy π such that $Y = X_T^{x,\pi}$ a. s.

Proposition

In a complete, arbitrage-free model, any claim Y has a unique arbitrage-free price $x = E_Q\left(e^{-rT} Y\right)$, where Q is the unique e. m. m.

Remark

- *In incomplete markets, there is, in general, no replicating portfolio, only super-replicating ones.*
- *Typically, there is an interval of arbitrage-free prices bounded by the super replication prices of the buyer and of the seller.*

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Black-Scholes PDE

- For simplicity, we pass to the discounted model by setting $\widetilde{S}^0_t = 1$ and $\widetilde{S}^i_t = e^{-rt} S^i_t$, $i = 1, \dots, n$.
- For a *European* contingent claim of the form $Y = f(S_T)$, let $C(t,x)$ denote the price of Y at time t given $S_t = x$, i. e. $C(t,x) = E_Q\left(e^{-r(T-t)} f(S_T) \middle| S_t = x\right)$.
- By the Feynman-Kac formula, C satisfies the PDE

$$\frac{\partial}{\partial t} C(t,x) + L_t C(t,x) = rC(t,x), \quad t \in [0,T], x \in \mathbb{R}^n,$$

with $C(T,x) = f(x)$. Here, the infinitesimal generator is

$$L_t g(x) = \sum_{i=1}^n r x^i \frac{\partial g}{\partial x^i}(x) + \frac{1}{2} \sum_{i,j=1}^n \left(\sigma \sigma^T\right)^{ij}(t,x) x^i x^j \frac{\partial^2 g}{\partial x^i \partial x^j}(x).$$

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Delta Hedging

Proposition (Delta Hedging)

The replicating portfolio for the discounted, European claim $e^{-rT}Y = e^{-rT}f(S_T)$ *is given by* $\pi^i_t = \frac{\partial C}{\partial x^i}(t, S_t)$, $i = 1, \dots, n$:

$$e^{-rT}Y = C(0, S_0) + \int_0^T \left\langle \nabla_x C(t, S_t), d\widetilde{S}_t \right\rangle .$$

Proof.

Apply Itô's formula to the process $e^{-rt}C(t, S_t)$. Note that we have to use the risk-neutral dynamics.

In particular, $\forall t \in [0, T] : X_t^{C(0,S_0),\pi} = e^{-rt}C(t, S_t)$. □

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In particular, $\forall t \in [0, T] : X_t^{C(0,S_0),\pi} = e^{-rt}C(t, S_t)$. □

Delta Hedging – 2

1. The derivative of the price with respect to the price of the underlying is called the *Delta* of a derivative, e. g. $\nabla_x C(t, x)$.
2. We can construct a self-financing portfolio as follows.

t	Y	$\widetilde{S}^0$	$\widetilde{S}^i$
0	-1	$C(0, S_0)$	0
t	-1	$C(t, S_t) - \left\langle \nabla_x C(t, S_t), \widetilde{S}_t \right\rangle$	$\frac{\partial C}{\partial x^i}(t, S_t)$
T	-1	Y	0

3. The term “Delta hedging” comes from the fact that this portfolio is *Delta neutral*, i. e. the Delta of the portfolio is 0. Note, however, that the portfolio requires continuous trading!

The Greeks

Definition

The derivatives of the price of a contingent claim with respect to model parameters are called the *Greeks*.

Remark

1. *The Greeks are generally used to "hedge against risks".*
2. *Some of the risks – like the "risk" of changing prices of the underlying (Delta hedging) – are inherent to the model.*
3. *Other risks are inherent to real-life restrictions: e. g., an investor implementing Delta hedging can only trade at discrete times. The corresponding risk can be countered by "Delta-Gamma-hedging".*
4. *There are also risks concerning changes of model parameters.*

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The Greeks – 2

Remark

In reality, hedging is usually very expensive. Therefore, the Greeks are rather used to monitor the development of a portfolio.

A non-exhaustive list of Greeks:

- Delta: derivative w. r. t. the price of the underlying.
- Gamma: second derivative w. r. t. the price of the underlying.
- Vega: derivative w. r. t. the volatility.
- Rho: derivative w. r. t. the interest rate.
- Theta: derivative w. r. t. time.

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Finite Differences

- In very few models like the classical Black-Scholes model explicit formulas for the option prices exist. Of course, these formulas can be differentiated to get formulas for the Greeks.
- Numerical differentiation is one method for calculation of the Greeks. Let $u(\alpha)$ denote the dependence of the price u of a derivative on some parameter α and choose $\epsilon > 0$ small enough. Use
$$\frac{u(\alpha+\epsilon)-u(\alpha)}{\epsilon}$$
as an approximation of $\frac{du}{d\alpha}$.
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The Logarithmic Trick

Given a family of random variables X^α, $\alpha \in \mathbb{R}$, having densities $p(\alpha, x)$, $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$, C^1 in α, and a bounded, measurable function f.

$$\begin{aligned}
\frac{d}{d\alpha} E(f(X^\alpha)) &= \frac{d}{d\alpha} \int_{\mathbb{R}} f(x) p(\alpha, x) dx \\
&= \int_{\mathbb{R}} f(x) \frac{\frac{\partial p}{\partial \alpha}(\alpha, x)}{p(\alpha, x)} p(\alpha, x) dx \\
&= \int_{\mathbb{R}} f(x) \frac{\partial \log(p(\alpha, x))}{\partial \alpha} p(\alpha, x) dx = E\big(f(X^\alpha)\pi^\alpha\big),
\end{aligned}$$

where $\pi^\alpha = \frac{\partial}{\partial \alpha} \log(p(\alpha, X^\alpha))$ does not depend on f.

The First Variation Process

Let X_t^x, $x \in \mathbb{R}^n$, $t \in [0, T]$, be the solution of the SDE

$$dX_t^x = a(X_t^x)dt + \sigma(X_t^x)dB^t \tag{1}$$

with $X_0^x = x$. Here, $a : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are C^1 functions with linear growth.
The *first variation process* is the $n \times n$-dimensional process given by

$$dJ_{0\to t}(x) = da(X_t^x) \cdot J_{0\to t}(x)dt + \sum_{i=1}^{d} d\sigma^i(X_t^x) \cdot J_{0\to t}(x)dB_t^i, \tag{2}$$

and $J_{0\to 0}(x) = I_n$, where σ^i denotes the ith column of σ.

Properties of the First Variation

1. If the coefficients of the SDE (1) are $C^{1+\epsilon}$, then there is a version of the solution X_t^x which is differentiable in x. In this case, the first variation is its Jacobian, i. e.

$$J_{0\to t}(x) = d_x X_t^x.$$

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The Malliavin Derivative

- Without getting into details, we present the formal set-up of Malliavin calculus.
- The *Malliavin derivative* is a closed operator $D : \mathcal{D}(D) \subset L^2(\Omega, \mathcal{F}, P) \to L^2([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}, dt \otimes P; \mathbb{R}^d)$. We write $D_s F(\omega) = DF(s, \omega)$, $F \in \mathcal{D}(D)$.
- The dual map δ is called *Skorohod stochastic integral*, i. e. $\delta : \mathcal{D}(\delta) \subset L^2([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}, dt \otimes P; \mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, P)$ satisfies

$$E(F\delta(u)) = E\Big(\int_0^T \langle D_s F, u_s \rangle_{\mathbb{R}^d} \, ds\Big), \tag{3}$$

where $F \in \mathcal{D}(D) \subset L^2(\Omega)$ and $u \in \mathcal{D}(\delta) \subset L^2([0,T] \times \Omega)$.

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The Malliavin Derivative – 2

Remark

Usually, calculation of Malliavin derivatives or Skorohod integrals is a hard problem. There are, however, some important special cases.

1. *For the process X^x solution of (1), the Malliavin derivative is given by*
$$D_s X_t^x = J_{0\to t}(x) J_{0\to s}(x)^{-1} \sigma(X_s^x) \mathbf{1}_{[0,t]}(s). \qquad (4)$$
2. *For a predictable process u, the Skorohod integral coincides with the Itô integral.*
3. *Let $F = (F^1, \dots, F^m)$, $F^i \in \mathcal{D}(D)$, and $\varphi \in C^1(\mathbb{R}^m)$, then $\varphi(F) \in \mathcal{D}(D)$ and $D\varphi(F) = \langle \nabla\varphi(F), DF \rangle_{\mathbb{R}^m}$.*

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Malliavin Weights

Definition

For a given stochastic process X_t^x, $t \in [0, T]$, as in (1) and a fixed time t, a *Malliavin weight* is a (sufficiently regular) random variable π such that

$$\nabla_x E(f(X_t^x)) = E(f(X_t^x)\pi) \tag{5}$$

for all, say, bounded, measurable $f : \mathbb{R}^n \to \mathbb{R}$.

Remark

- *By the "logarithmic trick", Malliavin weights exist for all hypo-elliptic diffusions.*
- *For simplicity, we concentrate on the Delta.*

Bismut-Elworthy-Li Formula

Theorem

Assume that $\sigma(X_t^x(\omega))$ has a right-inverse $R_t(\omega) \in \mathbb{R}^{d\times n}$ for $dt \otimes P$ a. e. (t,ω) such that $R_t J_{0\to t}(x)^i \in L^2([0,T]\times\Omega)$, $i = 1,\dots,n$, where $J_{0\to t}(x)^i$ denotes the ith column of $J_{0\to t}(x)$. Then for every bounded, measurable $f : \mathbb{R}^n \to \mathbb{R}$,

$$\nabla_x E(f(X_T^x)) = E\left(f(X_T^x)\int_0^T \frac{1}{T} R_t J_{0\to t}(x) dB_t\right). \tag{6}$$

Remark

The assumption is satisfied (with $R_s = \sigma^{-1}(X_s^x)$) if $d = n$, and σ is uniformly elliptic, i. e. $\exists \epsilon > 0$ s. t.

$$\xi^T \sigma(x)^T \sigma(x) \xi \geq \epsilon \left|\xi\right|^2, \quad \forall x, \xi \in \mathbb{R}^n.$$

Proof of the Bismut-Elworthy-Li Formula

$$\nabla_x E(f(X_T^x)) =$$
$$E\Big(\frac{1}{T}\int_0^T \nabla f(X_T^x)^T J_{0\to T}(x)\,\underbrace{J_{0\to t}(x)^{-1}\sigma(X_t^x)R_t J_{0\to t}(x)}_{=I_n}\,dt\Big)$$

The chain rule for Malliavin derivatives implies

$$D_t f(X_T^x) = \nabla f(X_T^x)^T D_t X_T^x = \nabla f(X_T^x)^T J_{0\to T}(x) J_{0\to t}(x)^{-1}\sigma(X_t^x)$$

and we get

$$\nabla_x E(f(X_T^x)) = E\Big(\int_0^T \langle D_t f(X_T^x), R_t J_{0\to t}(x)/T\rangle\, dt\Big)$$
$$= E(f(X_T^x)\delta(t \mapsto R_t J_{0\to t}(x)/T))$$

Bismut-Elworthy-Li Formula for a Parameter

Theorem

Let a, σ depend on a real parameter α and assume that σ is uniformly elliptic (in particular, $d = n$). Then

$$\frac{\partial}{\partial \alpha} E(f(X_T^x)) = E(f(X_T^x)\delta(t \mapsto H(t, T, \alpha))), \tag{7}$$

with $H(t, T, \alpha) = \frac{1}{T}\sigma(X_t^x, \alpha)^{-1} J_{0\to t}(x) J_{0\to T}(x)^{-1} \frac{\partial X_T^x}{\partial \alpha}$.

Proof.

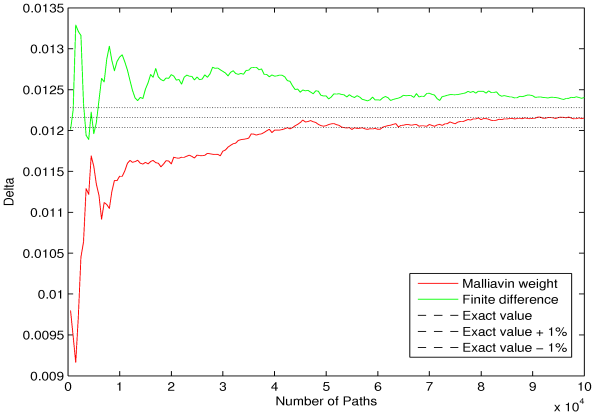
Proceed similarly to the first proof using

$$\nabla f(X_T^x)^T = D_t f(X_T^x)\sigma(X_t^x, \alpha)^{-1} J_{0\to t}(x) J_{0\to T}(x)^{-1}. \qquad \square$$

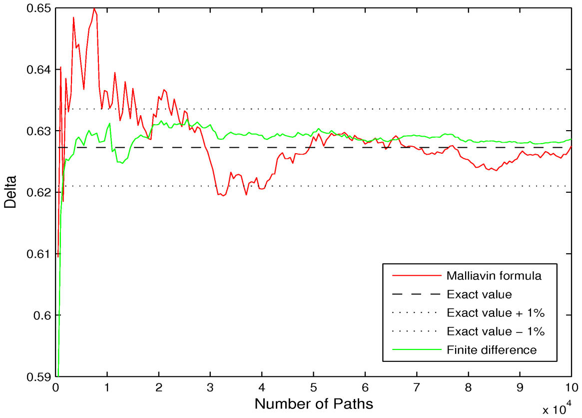
Remarks on the Bismut-Elworthy-Li Formula

- Similar formulas are also possible in a hypo-elliptic set up.
- In general, the Malliavin weight is defined as Skorohod integral of some process. Note that – unlike the Itô integral – the definition of the Skorohod integral is essentially non-constructive.
- Consequently, formulas for Malliavin weights especially in non-elliptic situations are often not directly usable for computational purposes.

Example: Delta of a Digital in the Bates Model



Example: Delta of a Call in the Merton Model



Summary

- The Greeks, sensitivities of option prices with respect to model parameters, are not only important for computational issues, but also because of their rôle in hedging strategies.
- Calculation using finite differences is often inefficient.
- Malliavin weights provide an alternative, in many situations superior method. In general, they are only given in a non-constructive way.
- Extending the theory to more general models, e. g. models driven by jump-diffusions, is a popular research topic.

References

- Jean-Michel Bismut: *Large deviations and the Malliavin calculus*, Birkäuser Verlag, 1984.
- Eric Fournié, Jean-Michel Lasry, Jérôme Lebuchoux, Pierre-Louis Lions, Nizar Touzi: *Applications of Malliavin calculus to Monte Carlo methods in finance*, Finance and Stochastics 3, 391–412, 1999.
- Ioannis Karatzas: *Lectures on the mathematics of finance*, CRM Monograph Series 8, AMS, 1996.
- David Nualart: *The Malliavin calculus and related topics*, Probability and Its Applications, Springer-Verlag, 1995.

Part II

Approximation of Malliavin Weights

Rewriting the Equation

We rewrite equation (1) as

$$dX_t^x = V_0(X_t^x)dt + \sum_{i=1}^{d} V_i(X_t^x) \circ dB_t^i = \sum_{i=0}^{d} V_i(X_t^x) \circ dB_t^i, \qquad (8)$$

where $\circ dB_t^i$ denotes the *Stratonovich stochastic integral* and we use the notation "$\circ dB_t^0 = dt$". $V_i(x) = \sigma^i(x)$, $i = 1, \dots, d$ and $V_0(x) = a(x) - \frac{1}{2}\sum_{i=1}^{d} dV_i(x) \cdot V_i(x)$.

Remark

We understand vector fields $V : \mathbb{R}^n \to \mathbb{R}^n$ as functions and as differential operators acting on $C^\infty(\mathbb{R}^n;\mathbb{R})$ by

$$Vf(x) = df(x) \cdot V(x), \quad f \in C^\infty(\mathbb{R}^n;\mathbb{R}),\ x \in \mathbb{R}^n.$$

Assumptions on the Data

- The vector fields $V_0, \ldots, V_d$ are smooth and C^∞-bounded, i. e. their derivatives of order greater than 0 are bounded.
- The vector fields satisfy a uniform Hörmander condition, see [Kusuoka 2002] for details. Consequently, the corresponding diffusion X is hypo-elliptic.

Stochastic Taylor Expansion

We define a degree on the set of multi-indices $\mathcal{A} = \bigcup_{k\in\mathbb{N}} \{0, \dots, d\}^k$ by $\deg((i_1, \dots, i_k)) = k + \#\{j | i_j = 0\}$, i. e. 0s are counted twice.

Theorem

Fix $m \in \mathbb{N}$ and $f \in C^\infty(\mathbb{R}^n)$. Then

$$f(X^x_t) = \sum_{\substack{\alpha=(i_1,\dots,i_k)\in\mathcal{A} \\ \deg(\alpha)\le m}} V_{i_1} \cdots V_{i_k} f(x) \int_{0<t_1<\cdots<t_k<t} \circ dB^{i_1}_{t_1} \circ \cdots \circ dB^{i_k}_{t_k} + R_m(f, t, x),$$

with $\sqrt{E(R_m(f, t, x)^2)} = \mathcal{O}(t^{(m+1)/2})$.

Proof of the Stochastic Taylor Expansion

Proof.

By Itô's formula, we get

$$f(X_t^x) = f(x) + \sum_{i=0}^{d} \int_0^t V_i f(X_s^x) \circ dB_s^i.$$

Now we apply Itô's formula to $V_i f(X_s^x)$, $i = 0, \dots, d$, and obtain

$$f(X_t^x) = f(x) + \sum_{i=0}^{d} \int_0^t \left(V_i f(x) + \sum_{j=0}^{d} \int_0^s V_j V_i f(X_u^x) \circ dB_u^j \right) \circ dB_s^i.$$

Iterate this procedure and then apply Itô's lemma to get the order estimate for the rest term. □

Remark

1. *The stochastic Taylor expansion is the starting point for many applications in stochastic analysis and for several numerical methods, including*
    - *Stochastic Taylor schemes and*
    - *Cubature on Wiener space, a method based on Terry Lyon's theory of rough paths.*
2. *Following the latter concept, we interpret the process* $(B^\alpha_t)_{\alpha\in\mathcal{A},\,\deg(\alpha)\leq m}$, $t \in [0, T]$, *as the "probabilistic core" of the solution of the SDE.*
3. *Consequently, a better understanding of this process might yield methods to calculate the non-constructive Malliavin-weight formulas presented in the last section.*

Passing to the Algebraic Framework

- In order to study the stochastic process given by the iterated Stratonovich integrals up to order m, we embed it into an appropriate algebraic/geometric framework.
- We additionally assume that the Stratonovich drift vanishes, i. e. $V_0 = 0$. This assumption simplifies the notation and allows us to use the usual degree $\deg((i_1, \dots, i_k)) = k$ for $(i_1, \dots, i_k) \in \{1, \dots, d\}^k$. Furthermore, it will allow us to omit some subtle restrictions in the following.
- We stress that the results remain *essentially* true for $V_0 \neq 0$.

The Free, Nilpotent Algebra

- Let $\mathbb{A}^m_{d,0}$ be the space of all non-commutative polynomials in $e_1, \ldots, e_d$ of degree less than or equal to m.
- Define a non-commutative multiplication on $\mathbb{A}^m_{d,0}$ by cutting off all monomials of higher degree than m. $\mathbb{A}^m_{d,0}$ becomes the *free associative, non-commutative, step-m nilpotent real algebra with unit* in d generators $e_1, \ldots, e_d$.
- Let W_0 denote the linear span of the unit element 1 of $\mathbb{A}^m_{d,0}$, i. e. W_0 is the space of all polynomials of degree 0. We identify $W_0 \simeq \mathbb{R}$ and denote by x_0 the projection of $x \in \mathbb{A}^m_{d,0}$ on W_0.

The Free, Nilpotent Algebra – 2

- $\exp : \mathbb{A}^m_{d,0} \to \mathbb{A}^m_{d,0}$ is defined by $\exp(x) = 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!}$.
- The logarithm is defined for $x_0 \neq 0$ by

$$\log(x) = \log(x_0) + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \left(\frac{x - x_0}{x_0} \right)^i$$
$$= \log(x_0) + \sum_{i=1}^{m} \frac{(-1)^{i-1}}{i} \left(\frac{x - x_0}{x_0} \right)^i.$$

- $\mathbb{A}^m_{d,0}$ equipped with the commutator bracket $[x, y] = xy - yx$, $x, y \in \mathbb{A}^m_{d,0}$, is a Lie algebra.

The Free, Nilpotent Lie Group

- Let $\mathfrak{g}^m_{d,0}$ denote the *free, step-m nilpotent Lie algebra* generated by $\{e_1, \dots, e_d\}$, i. e.

$$\mathfrak{g}^m_{d,0} = \left\langle \{e_i, [e_i, e_j], [e_i, [e_j, e_k]], \dots \mid i, j, k = 1, \dots, d\} \right\rangle.$$

- We define the *step-m nilpotent free Lie group* as the exponential image of $\mathfrak{g}^m_{d,0}$, i. e. $G^m_{d,0} = \exp(\mathfrak{g}^m_{d,0})$.
- $G^m_{d,0}$ is a Lie group and $\mathfrak{g}^m_{d,0}$ is its Lie algebra, which can be seen by the Campbell-Baker-Hausdorff formula

$$\exp(x)\exp(y) = \exp\left(x+y+\frac{1}{2}[x,y]+\frac{1}{12}([x,[x,y]]-[y,[y,x]])+\cdots\right).$$

Note that $\exp : \mathfrak{g}^m_{d,0} \to G^m_{d,0}$ and $\log : G^m_{d,0} \to \mathfrak{g}^m_{d,0}$ define a global chart of the Lie group.

Example: The Heisenberg Group

- The Heisenberg group is the group $G^2_{2,0}$, a 3-dimensional submanifold of $\mathbb{A}^2_{2,0} \simeq \mathbb{R}^7$.
- It allows the following representation as a matrix group:

$$G^2_{2,0} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

The Lie algebra – as tangent space at $I_3 \in G^2_{2,0}$ – is

$$\mathfrak{g}^2_{2,0} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

- An isomorphism is given by $e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Encoding Iterated Stratonovich Integrals

Definition

For $y \in \mathbb{A}^m_{d,0}$ we define a stochastic process Y^y_t, $t \in [0, T]$, by

$$Y^y_t = y\Big(\sum_{\alpha\in\mathcal{A},\ \deg(\alpha)\le m} B^\alpha_t e_\alpha\Big),$$

where $e_\alpha = e_{i_1} \cdots e_{i_k}$ for $\alpha = (i_1, \ldots, i_k)$, $e_\emptyset = 1$, and B^α_t denotes the corresponding iterated Stratonovich integral, i. e. $B^\alpha_t = \int_{0<t_1<\cdots<t_k<t} \circ dB^{i_1}_{t_1} \cdots \circ dB^{i_k}_{t_k}$.

Remark

Y^1_t is the vector of the iterated Stratonovich integrals written in the canonical basis of $\mathbb{A}^m_{d,0}$.

The Geometry of the Iterated Integrals

Theorem

Define vector fields $D_i(x) = xe_i$, $x \in \mathbb{A}^m_{d,0}$, $i = 1, \dots, d$.

1. *Y^y_t is solution of the SDE*

$$dY^y_t = \sum_{i=1}^d D_i(Y^y_t) \circ dB^i_t, \quad Y^y_0 = y.$$

2. *Given $y \in G^m_{d,0}$, we have $Y^y_t \in G^m_{d,0}$ a. s. for all $t \in [0, T]$.*
3. *By the Feynman-Kac formula,*

$$E(Y^y_t) = y \exp\Big(\frac{t}{2} \sum_{i=1}^d e_i^2\Big).$$

The Geometry of the Iterated Integrals

Theorem

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3. *By the Feynman-Kac formula,*

$$E(Y^y_t) = y \exp\left(\frac{t}{2} \sum\nolimits_{i=1}^d e_i^2\right).$$

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2. *Given $y \in G^m_{d,0}$, we have $Y^y_t \in G^m_{d,0}$ a. s. for all $t \in [0, T]$.*
3. *By the Feynman-Kac formula,*
$$E(Y^y_t) = y \exp\Big(\frac{t}{2} \sum_{i=1}^d e_i^2\Big).$$

The Geometry of the Iterated Integrals – 2

Remark

1. *The formula (3) allows efficient calculation of all moments of iterated Stratonovich integrals. Observe that $E(Y^1_t) \notin G^m_{d,0}$.*
2. *By Statement 2, we can carry out all relevant calculations for iterated Stratonovich integrals in the vector space $\mathfrak{g}^m_{d,0}$ by passing to the process $Z_t = \log(Y^1_t)$.*

Example

For $m = d = 2$, we get $Z_t = B^1_t e_1 + B^2_t e_2 + A_t[e_1, e_2]$, where A_t denotes Lévy's area

$$A_t = \frac{1}{2}\int_0^t B^1_s \circ dB^2_s - \frac{1}{2}\int_0^t B^2_s \circ dB^1_s.$$

Malliavin Weights on $G^m_{d,0}$

Proposition

Fix $w \in \mathfrak{g}^m_{d,0}$ and $t \in [0, T]$. There is a non-adapted, Skorohod integrable, $\mathbb{R}^d$-valued process a_s, $0 \leq s \leq T$ such that for any bounded, measurable function $f : G^m_{d,0} \to \mathbb{R}$ and any $y \in G^m_{d,1}$

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} E(f(Y_t^{y+\epsilon w})) = E(f(Y_t^y)\pi^m_{d,0}),$$

where $\pi^m_{d,0} = \delta(a)$.

Proof.

The proof is very similar to the proof for the existence of Malliavin weights in a general, hypo-elliptic setting. See [Teichmann 06]. □

Malliavin Weights on $G^m_{d,0}$ – 2

Remark

1. *The strategy a is universal in the sense that it does only depend on m, d, t and – in a linear way – w.*
2. *Yet again, the Malliavin weight is given in an essentially non-constructive way due to the Skorohod integration. Note that even calculation of the strategy a is difficult. However, approximation of the weight $\pi^m_{d,0}$ turns out to be possible.*

Approximation with Universal Weights

Theorem

Fix $x, v \in \mathbb{R}^n$, $t \in [0, T]$ and $m \geq 1$ such that v can be written as

$$v = \sum_{\alpha \in \mathcal{A} \setminus \{\emptyset\},\ \deg(\alpha) \leq m-1} w_\alpha [V_{i_1}, [V_{i_2}, [\cdots, V_{i_k}] \cdots](x)$$

for some $w_\alpha \in \mathbb{R}$. Define $w = \sum w_\alpha [e_{i_1}, [e_{i_2}, [\cdots, e_{i_k}] \cdots] \in \mathfrak{g}^m_{d,0}$ and let $\pi^m_{d,0}$ denote the corresponding universal Malliavin weight. Then for any C^∞-bounded function $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = E(f(X_t^x)\pi^m_{d,0}) + \mathcal{O}(t^{(m+1)/2}).$$

We note that the constant in the leading order term in the error estimate depends only on the first derivative of f.

Cubature

Definition

Given a finite Borel measure μ on $\mathbb{R}^n$ with finite moments of order up to $m \in \mathbb{N}$. A *cubature formula* of degree m is a collection of weights $\lambda_1, \dots, \lambda_k > 0$ and points $x_1, \dots, x_k \in \operatorname{supp}(\mu) \subset \mathbb{R}^n$ such that the following equality holds for any polynomial f on $\mathbb{R}^n$ of degree less or equal m:

$$\int_{\mathbb{R}^n} f(x)\mu(dx) = \sum_{i=1}^{k} \lambda_i f(x_i).$$

Chakalov's Theorem

Theorem

For any Borel measure μ on $\mathbb{R}^n$ with finite moments of order up to m there is a cubature formula with size $k \leq \dim \mathrm{Pol}_m(\mathbb{R}^n)$, the space of polynomials on $\mathbb{R}^n$ of order up to m.

Remark

- *Chakalov's Theorem is non-constructive, and construction of efficient cubature formulas remains a non-trivial problem in higher dimensions.*
- *The bound on the size is a consequence of Caratheodory's Theorem.*

Cubature Formulas for the Universal Weight

Theorem

Fix $t \in [0, T]$ and $w \in \mathfrak{g}^m_{d,0}$. There are points $x_1, \ldots, x_r \in G^m_{d,0}$ and weights $\rho_1, \ldots, \rho_r \neq 0$ such that

$$E(Y^1_t \pi^m_{d,0}) = w \exp\Big(\frac{t}{2}\sum_{i=1}^d e_i^2\Big) = \sum_{j=1}^r \rho_j x_j.$$

Furthermore, we may choose $r \leq 2 \dim \mathbb{A}^m_{d,0} + 2$.

Proof of Cubature for the Universal Weight

Proof.

The first equality is an easy consequence of

$$E(Y_t^y) = y \exp\Big(\frac{t}{2} \sum\nolimits_{i=1}^d e_i^2\Big), \quad y \in \mathbb{A}_{d,0}^m.$$

Now define two positive measures on the Wiener space by $\frac{dQ_+}{dP} = (\pi_{d,0}^m)_+$ and $\frac{dQ_-}{dP} = (\pi_{d,0}^m)_-$. By absolute continuity of $Q_\pm$ w. r. t. P, $Y_t^1 \in G_{d,0}^m$ $Q_\pm$-a. s.. Chakalov's Theorem applied to the laws of Y_t^1 under Q_+ and Q_- yields

$$E_P(Y_t^1 \pi_{d,0}^m) = E_{Q_+}(Y_t^1) - E_{Q_-}(Y_t^1) = \sum\nolimits_{j=1}^r \rho_j x_j. \qquad \square$$

A Reformulation of Cubature

- A theorem from sub-Riemannian geometry (Chow's Theorem) implies that any $x \in G^m_{d,0}$ can be joined to 1 by the the solution of the ODE
$$\dot{x}_t = \sum_{i=1}^{d} D_i(x_t)\dot{\omega}_i(t) = \sum_{i=1}^{d} x_t e_i \dot{\omega}_i(t)$$
for some paths of bounded variation $\omega_i : [0, T] \to \mathbb{R}$, $i = 1, \dots, d$, i. e. $x_0 = 1$ and $x_t = x$.
- We denote the solution to the above ODE for some path ω of bounded variation by $Y^1_s(\omega)$, $s \in [0, T]$.
- Consequently, there are $\omega_j \in C_{bv}([0, T]; \mathbb{R}^d)$, $j = 1, \dots, r$, s. t.
$$E(Y^1_t \pi^m_{d,0}) = \sum_{j=1}^{r} \rho_j Y^1_t(\omega_j).$$

Malliavin Weights by Cubature

Theorem

Fix $x, v \in \mathbb{R}^n$, $t \in [0, T]$ and $m \geq 1$ such that v can be written as

$$v = \sum_{\alpha \in \mathcal{A} \setminus \{\emptyset\}, \ \deg(\alpha) \leq m-1} w_\alpha [V_{i_1}, [V_{i_2}, [\cdots, V_{i_k}] \cdots](x)$$

for some $w_\alpha \in \mathbb{R}$. Define $w = \sum w_\alpha [e_{i_1}, [e_{i_2}, [\cdots, e_{i_k}] \cdots] \in \mathfrak{g}^m_{d,0}$ and let ρ_j, ω_j, $j = 1, \ldots, r$, denote the cubature formula for the corresponding weight $\pi^m_{d,0}$. Then

$$\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = \sum_{j=1}^{r} \rho_j f(X_t^x(\omega_j)) + \mathcal{O}(t^{(m+1)/2}),$$

where $\frac{dX_t^x(\omega_j)}{dt} = \sum_{i=1}^d V_i(X_t^x(\omega_j))\dot{\omega}_j^i(t)$ with $X_0^x(\omega_j) = x$.

Remarks

- The above formula is some kind of finite difference formula! Instead of solving the PDE problem for different starting points, we solve the SDE problem for different trajectories ω of the Brownian motion.
- The same procedure *without* $\pi^m_{d,0}$ gives a method for calculation of $E(f(X^x_t)) \approx \sum_{j=1}^{l} \lambda_j f(X^x_t(\tilde{\omega}_j))$. This method – introduced by T. Lyons and N. Victoir – is known as "Cubature on Wiener space".
- For actual computations, it is necessary to iterate the procedure along a partition $0 = t_0 < t_1 < \cdots < t_k = t$ of $[0, t]$. This is possible using cubature on Wiener space, yielding, at least in theory, a method of order $\frac{m-1}{2}$ for approximation of Greeks.

The Heat Kernel on $G^m_{d,0}$

Definition

The density $p_t(x)$, $x \in G^m_{d,0}$, of the process Y^1_t with respect to the Haar measure on the Lie group is called *heat kernel*.

Remark

1. *The name comes from the fact that* p_t *is the fundamental solution of the heat equation with respect to the sub-Laplacian* $L = \frac{1}{2}\sum_{i=1}^{d} D_i^2$*, the infinitesimal generator of* Y^1_t*.*
2. *We can equivalently study the density of* $Z_t = \log(Y^1_t)$ *with respect to the Lebesgue measure on* $\mathfrak{g}^m_{d,0}$*.*
3. *In the setting without drift, the heat kernel always exists as a Schwartz function. With non-vanishing drift, we need to factor out the direction in* $\mathfrak{g}^m_{d,1}$ *corresponding to* t*.*

Approximation of the Heat Kernel

- We want to find polynomial approximations of the heat kernel p_t or the Malliavin weight $d \log p_t \circ Y_t^1$. In the following we concentrate on the former problem and tacitly switch to $\mathfrak{g}_{d,0}^m$ using the same notation p_t for the density of Z_t on $\mathfrak{g}_{d,0}^m$.
- Choose a suitable (Gaussian) measure Q_t with density r_t on $\mathfrak{g}_{d,0}^m$ and let $h_\alpha(t, \cdot)$ denote the corresponding family of orthonormal (Hermite) polynomials.
- We need to calculate the integral

$$\int_{\mathfrak{g}_{d,0}^m} \frac{p_t(z)}{r_t(z)} h_\alpha(t, z) Q_t(dz) = \int_{\mathfrak{g}_{d,0}^m} p_t(z) h_\alpha(t, z) dz = E(h_\alpha(t, Z_t)).$$

Approximation of the Heat Kernel – 2

- This procedure will give us an approximation

$$\frac{p_t(z)}{r_t(z)} = \sum_{\alpha} a_t^{\alpha} h_{\alpha}(t,z)$$

valid in $L^2(\mathfrak{g}^m_{d,0}, Q_t)$, where $a_t^{\alpha} = E(h_{\alpha}(t, Z_t))$.
- Note that $h_{\alpha}(t, Z_t)$ is a (time dependent) *polynomial* in the iterated Stratonovich integrals of order up to m. Calculation of a_t^{α} is enabled by

$$E(\tilde{Y}_t^1) = \exp_{\tilde{m}}\Big(\frac{t}{2}\sum\nolimits_{i=1}^{d} e_i^2\Big),$$

where $\tilde{Y}$ denotes the process of iterated integrals in $\mathbb{A}^{\tilde{m}}_{d,0}$ with $\tilde{m} > m$ large enough.
- The use of Q_t instead of dz is preferable because polynomials are not dz-square integrable.

Applications of Approximate Heat Kernels

- Approximate heat kernels can be used to make higher order Taylor schemes for approximation of SDEs feasible.
- In the context of Mallivin weights, they provide approximations to the universal Malliavin weights defined before.
- Finally, the subject of heat kernels on Lie groups is a well-established subject of mathematical research in its own right.

Summary

- By using Stochastic Taylor expansion, we constructed universal Malliavin weights, which yield approximations to the Greeks for a very general class of (d-dimensional) problems.
- Cubature formulas for Greeks – in combination with cubature on Wiener space – yield high-order methods for the calculation of the Greeks, even in situations, where no direct formula for the Greeks is possible.
- There are still many open problems regarding actual usability of this theory for computations.

References

- Shigeo Kusuoka: *Appropximation of expectation of diffusion processes based on Lie algebra and Malliavin calculus*, Advances in mathematical economics 6, 69-83, 2004.
- Terry Lyons: *Differential equations driven by rough signals*, Rev. Mat. Iberoam. 14, No.2, 215-310, 1998.
- Terry Lyons and Nicolas Victoir: *Cubature on Wiener space*, Proceedings of the Royal Society London A 460, 169-198, 2004.
- Josef Teichmann: *Calculating the Greeks by Cubature formulas*, Proceedings of the Royal Society London A 462, 647-670, 2006.