Cubature on Wiener space for infinite-dimensional SDEs

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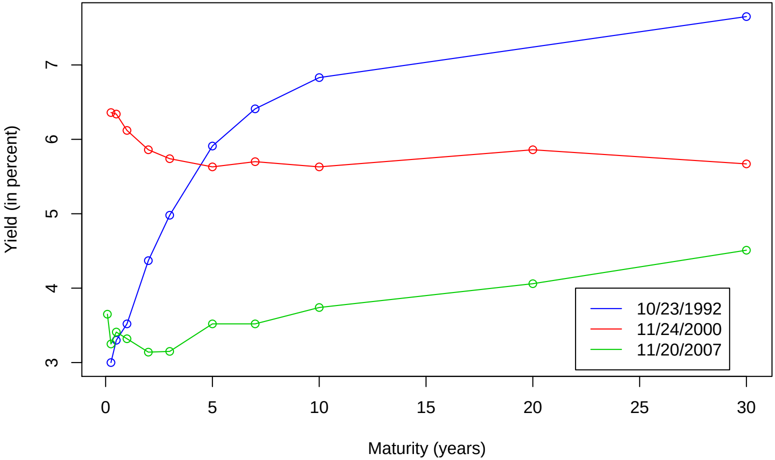
Outline

Some US Treasury yield curves



Forward rates

Definition

Let $P(t,T)$ denote the price at time t of a zero coupon bond with maturity $T \geq t$. Then the *instantaneous forward rate* at time t for maturity T is defined by

$$f(t,T) = -\frac{\partial}{\partial T} \log P(t,T).$$

$f(t,t)$ is called *spot rate* or *short rate*.

- $P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$.
- Important for pricing and valuation of interest rate products

Modeling interest rates

All models understood under the risk-neutral measure

Short rate models: if the short rate $f(t,t)$ is modeled, then the term structure is constructed by

$$P(t,T) = E\Big(\exp\Big(-\int_t^T f(s,s)ds\Big)\Big|\mathcal{F}_t\Big),$$

e.g., Vasiček, Hull-White, Cox-Ingersoll-Ross model.

Libor Market Model: simultaneous modeling of several yields

Heath-Jarrow-Morton framework: simultaneous modeling of the whole forward rate curve $(f(t,T))_{T\in[t,\infty[}$, $t \in [0,\infty[$

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HJM framework – 1

- HJM: model for time evolution of the forward rate curve $(f(t,T))_{T\in[t,\infty[}$, $t \in [0,\infty[$
- $r_t(x) := f(t, t+x)$, $x \in [0,\infty[$, $t \in [0,\infty[$, (Musiela parametrization)
- H is a suitable (separable, real) Hilbert space of (forward rate) curves, in fact it is a weighted Sobolev space.
- Model driven by finite number of Brownian motions: $B_t = (B_t^1, \ldots, B_t^d)_{t\in[0,\infty[}$ (HJM also possible for infinite-dimensional BM)

HJM framework – 2

$$dr_t = \left(\frac{\partial}{\partial x} r_t + \alpha_{HJM}(r_t)\right)dt + \sum_{i=1}^{d} \sigma_i(r_t) dB_t^i \qquad (1)$$

- $r_0 \in H$, initial forward rate curve
- $\sigma_i : H \to H$, $i = 1, \ldots, d$, volatility vector fields (may be time-dependent)
- $\alpha_{HJM} : H \to H$ drift vector field given by

$$\alpha_{HJM}(h)(x) = \sum_{i=1}^{d} \sigma_i(h)(x) \int_0^x \sigma_i(h)(y)dy$$

HJM framework – 2

$$dr_t = \Big(\underbrace{\frac{\partial}{\partial x} r_t}_{\text{Musiela-param.}} + \underbrace{\alpha_{HJM}(r_t)}_{\text{No Arbitrage}} \Big) dt + \sum_{i=1}^{d} \underbrace{\sigma_i(r_t)}_{\text{parameter}} dB_t^i$$

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Pricing in HJM

The price of an interest rate option with payoff $f : H \to \mathbb{R}$ is given by

$$c(t, T) = E(f(r_T)/B_{t,T}|\mathcal{F}_t).$$

- Requires weak approximation of the solution of the HJM-equation
- Known methods only in the Gaussian case and in the case of finite-dimensional realizations
- Scenario simulation necessary for risk analysis

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SDEs in infinite dimensions

Consider

$$\begin{cases} dX_t^x = (AX_t^x + \alpha(X_t^x))dt + \sum_{i=1}^{d} \sigma_i(X_t^x)dB_t^i, \\ X_0^x = x \in H, \end{cases} \tag{2}$$

where

- B is a d-dimensional Brownian motion,
- H is a separable, real Hilbert space and

Assumption A

$A : \mathcal{D}(A) \subset H \to H$ is the generator of a C_0-semigroup $(S_t)_{t \in [0,\infty[}$ of operators on H and $\alpha, \sigma_1, \dots, \sigma_d : H \to H$ are C^∞-bounded vector fields.

Mild solutions

Definition

A continuous H-valued process X_t^x is a *mild solution* of the SDE (2) if

$$X_t^x = S_t x + \int_0^t S_{t-s}\alpha(X_s^x)ds + \sum_{i=1}^d \int_0^t S_{t-s}\sigma_i(X_s^x)dB_s^i.$$

- Mild solutions are not necessarily semi-martingales.
- Each strong solution is a mild solution by variation of constants.

ODEs along cubature paths

Given an N-dimensional SDE

$$dX_t^x = \sum_{i=0}^{d} V_i(X_t^x) \circ dB_t^i = V_0(X_t^x)dt + \sum_{i=1}^{d} V_i(X_t^x) \circ dB_t^i. \quad (3)$$

Fix a uniform partition of $[0, T]$ of size $\ell + 1$.

- $\omega_{j_1,\dots,j_\ell} : [0, T] \to \mathbb{R}^d$ is the path of bounded variation found by concatenating *cubature paths* $\omega_{j_r} : [0, T/\ell] \to \mathbb{R}^d$ (with weights λ_{j_r}).
- Given a path of bounded variation $\omega : [0, T] \to \mathbb{R}^d$, $X_T^x(\omega)$ denotes the solution of the ODE (3) with B formally replaced by ω.

Weak approximation

Theorem (Lyons and Victoir)

Given a smooth function $f : \mathbb{R}^N \to \mathbb{R}$ (bounded with bounded derivatives of order up to $m+1$), then

$$\sup_{x\in\mathbb{R}^N}\left| E(f(X_T^x)) - \sum_{(j_1,\dots,j_\ell)\in\{1,\dots,n\}^\ell} \lambda_{j_1}\cdots\lambda_{j_\ell} f(X_T^x(\omega_{j_1,\dots,j_\ell}))\right| \le CT\left(\frac{T}{\ell}\right)^{\frac{m-1}{2}}.$$

Cubature formulas on Wiener space

Definition

Positive weights $\lambda_1, \ldots, \lambda_n$ and paths $\omega_1, \ldots, \omega_n : [0, T] \to \mathbb{R}^d$ form a *cubature formula on Wiener space* of degree m if

$$E\Big(\int_{0<t_1<\cdots<t_k<T} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k}\Big) = \sum_{j=1}^{n} \lambda_j \int_{0<t_1<\cdots<t_k<T} d\omega_j^{i_1}(t_1) \cdots d\omega_j^{i_k}(t_k)$$

for all $J = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k$ with $\deg(J) \le m$, $k \ge 0$.

Convention: $B_t^0 = \omega_j^0(t) = t$, $\deg(i_1, \ldots, i_k) = k + \#\{j : i_j = 0\}$, i.e., zeros are counted twice.

Remarks

- Non-uniform partitions can accelerate convergence and improve differentiability assumptions
- As typical for weak schemes, precise integration usually not possible: use Monte-Carlo simulation
- Sophisticated recombination techniques available
- Respects the geometry of the problem (invariant submanifolds, support of the law)
- Existence proof relying on the geometry of iterated Stratonovich integrals, Chakalov's theorem, and Chow's theorem
- Introduced by Terry Lyons and Nicolas Victoir (2004); strongly related to *moment similar random variables* by Shigeo Kusuoka (2001)

Weak schemes in infinite dimensions

- Finite element schemes (reducing the problem to a stochastic equation on a finite dimensional subspace), e. g. (Hausenblas 2003)
- Only few results on finite difference schemes (Gyöngy 1998)
- No “general” theory available
- Usual Euler-Maruyama schemes do not fit with the concept of mild solutions.

Idea

Cubature scheme can be immediately generalized to the infinite dimensional situation, since the results do not depend on the dimension of the state space.

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PDEs along cubature paths

- Recall the SDE (2) in H

$$dX_t^x = (AX_t^x + \alpha(X_t^x))dt + \sum_{i=1}^{d} \sigma_i(X_t^x)dB_t^i.$$

- Define vector fields

$$\alpha_0(x) = \alpha(x) - \frac{1}{2}\sum_{i=1}^{d} D\sigma_i(x)\cdot\sigma_i(x), \quad x \in H,$$
$$\sigma_0(x) = Ax + \alpha_0(x), \quad x \in \mathcal{D}(A) \subset H.$$

- In general, there is no Stratonovich formulation of (2).
- For a fixed path $\omega : [0, T] \to \mathbb{R}^d$ of bounded variation and $x \in H$ let $X_t^x(\omega)$ be the solution of

$$X_t^x(\omega) = S_t x + \int_0^t S_{t-s}\alpha_0(X_s^x(\omega))ds + \sum_{i=1}^{d}\int_0^t S_{t-s}\sigma_i(X_s^x(\omega))d\omega^i(s).$$

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Sobolev spaces

- Define the Sobolev spaces by $\mathcal{D}(A^0) = H$ and

$$\mathcal{D}(A^{k+1}) = \{x \in \mathcal{D}(A^k) \mid Ax \in \mathcal{D}(A^k)\}$$

  with the graph norm

$$\|x\|^2_{\mathcal{D}(A^k)} = \|x\|^2_H + \sum_{i=1}^{k} \|A^i x\|^2_H .$$

- $\mathcal{D}(A^\infty) := \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$ is a Fréchet space with metric

$$d_{\mathcal{D}(A^\infty)}(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_{\mathcal{D}(A^n)}}{\max(1, \|x - y\|_{\mathcal{D}(A^n)})}$$

Hierarchy of Sobolev spaces

$$
\begin{tikzcd}
H \arrow[r, "S_t"] \arrow[d, "R(\lambda{,}A)"] & H \\
\mathcal{D}(A) \arrow[r, "S_t"] \arrow[d, "R(\lambda{,}A)"] & \mathcal{D}(A) \arrow[u, "\lambda - A"'] \\
\mathcal{D}(A^2) \arrow[r, "S_t"] \arrow[d, dotted, no head] & \mathcal{D}(A^2) \arrow[u, "\lambda - A"'] \arrow[d, dotted, no head] \\
\mathcal{D}(A^\infty) \arrow[r, "S_t"] & \mathcal{D}(A^\infty)
\end{tikzcd}
$$

- S_t can be restricted as semi-group to $\mathcal{D}(A^n)$, $1 \leq n \leq \infty$.
- A no longer unbounded on $\mathcal{D}(A^\infty)$, but Fréchet spaces not easy for studying ODEs.
- Go as far as necessary, but not further.

Some assumptions

Assumption B

$\alpha, \sigma_1, \ldots, \sigma_d$ are smooth vector fields mapping $\mathcal{D}(A^k) \to \mathcal{D}(A^k)$ and their restrictions to $\mathcal{D}(A^k)$ are C^∞-bounded as maps $\mathcal{D}(A^k) \to \mathcal{D}(A^k)$, $k \in \mathbb{N}$.

Assumption C

$f \in C^\infty(H; \mathbb{R})$ and $x \in \mathcal{D}(A^{\lfloor \frac{m}{2} \rfloor + 1})$ such that

$$\sup_{0 \le t \le T} \sup_{y \in \mathfrak{S}_T(x)} \left| \sigma_{i_1} \cdots \sigma_{i_k} P_t f(y) \right| < \infty$$

for each multi-index $(i_1, \ldots, i_k) \in \{0, \ldots, d\}^k$ with $m < \deg \le m + 2$, $k \in \mathbb{N}$.

Main result

Theorem

Given an operator A and vector fields $\alpha, \sigma_1, \ldots, \sigma_d$ satisfying Assumptions A and B, and a point x and a functional f satisfying Assumption C. Then

$$\left| E(f(X_T^x)) - \sum_{(j_1,\ldots,j_\ell)\in\{1,\ldots,n\}^\ell} \lambda_{j_1}\cdots\lambda_{j_\ell} f(X_T^x(\omega_{j_1,\ldots,j_\ell}))\right| \leq CT\left(\frac{T}{\ell}\right)^{\frac{m-1}{2}}.$$

Remarks

Idea of the proof

1. For $x \in \mathcal{D}(A)$, solve the SDE in both Hilbert spaces H and $\mathcal{D}(A)$
2. By uniqueness of solutions we are given a semimartingale in H.
3. Iterate this procedure for stochastic Taylor expansion.

- Weak method of any order with deterministic a-priori bounds
- Assumption B is not very restrictive in view of the HJM framework: indeed, one often has $\sigma_i = \phi_i \circ \mu_i$, where μ_i is a continuous linear map $H \to \mathbb{R}^p$ and $\phi_i : \mathbb{R}^p \to \mathcal{D}(A^\infty)$ is smooth, for some $p \geq 1$.
- Geometry of the problem respected (invariant submanifolds, support of the distribution)

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Remark on Assumption C

Remark

Assumption C can be realized by applying the isomorphism

$$R(\lambda, A)^{-\lfloor \frac{m}{2} \rfloor} : \mathcal{D}(A^{\lfloor \frac{m}{2} \rfloor}) \to H$$

and cutting off the vector fields in $\mathcal{D}(A^{\lfloor \frac{m}{2} \rfloor})$ outside a large set (with respect to $\|\cdot\|_{\mathcal{D}(A^{\lfloor \frac{m}{2} \rfloor})}$).

Note that the solution process only hits the complement of a ball with large radius in $\mathcal{D}(A^{\lfloor \frac{m}{2} \rfloor})$ with negligible probability by Lipschitz continuity of the driving vector fields on $\mathcal{D}(A^{\lfloor \frac{m}{2} \rfloor})$.

Cubature including jumps

- Consider infinite dimensional (finite-activity) jump diffusion

$$dX_t^x = (AX_t^x + \alpha(X_t^x))dt + \sigma(X_t^x)dB_t + \delta(X_t^x)dL_t,$$

  L_t a compound Poisson process with rate μ.
- $E\left(f(X_t^x)\right) = \sum_{n=0}^{\infty} \frac{\mu^n e^{-t\mu n}}{n!} t^n E\left(f(X_t^x) \middle| N_t = n\right)$
- Thus, can use a cubature formula on Wiener space of degree $m - 2n$ for approximation of $E\left(f(X_t^x) \middle| N_t = n\right)$
- Integration over jump-times and jump-sizes required

Method of the moving frame – a new proof method

- Difficulty in SPDE: *unboundedness of drift* and *noise*
- Separate treatment of drift and noise
- Assume that A is generator of a group $(S_t)_{t\in\mathbb{R}}$.
- Consider the Stratonovich SPDE

$$dX_t = (AX_t + \alpha(X_t))dt + \sum_{i=1}^{d} \sigma_i(X_t) \circ dB_t^i.$$

- Define $Y_t = S_{-t}X_t$. Then

$$dY_t = \widetilde{\alpha}(t, Y_t)dt + \sum_{i=1}^{d} \widetilde{\sigma}_i(t, Y_t) \circ dB_t^i,$$

with $\widetilde{\alpha}(t,x) = S_{-t}\alpha(S_t x)$, $\widetilde{\sigma}_i(t,x) = S_{-t}\sigma_i(S_t x)$.

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Cubature in the moving frame

1. Assume that modified vector fields $\widetilde{\alpha}, \widetilde{\sigma}_i$ are C^∞ bounded.
2. Stochastic Taylor expansion of Y_t precisely as in finite dimensions
3. Cubature on Wiener space under some additional boundedness condition (that is satisfied if the quantity of interest f and all vector fields have bounded support).
4. Transfer cubature method for Y_t to $X_t = S_t Y_t$
5. Leads to the same method as introduced before, but in a different manner.

The moving frame for pseudo-contractive semigroups

- $(S_t)_{t \geq 0}$ a *pseudo-contractive* semigroup on H, i.e.,

$$\|S_t\| \leq \exp(\omega t).$$

- *Szőkefalvi-Nagy theorem*: There is a Hilbert space $H \subset W$ and a C_0-group of bounded operators $(Q_t)_{t \in \mathbb{R}}$ thereon extending (S_t) in the sense that $S_t x = \pi Q_t x$ for $x \in H$.
- Extend driving vector-fields to W via π, e.g., $\alpha \to \alpha \circ \pi$.
- Check smoothness and boundedness conditions on W.

A remark on computation of the weighted sum

- Approximate the summation over the cubature tree via Monte-Carlo simulation.
- Alternatively: Use recombination method.
  - Consider group generated by $\gamma_1, \ldots, \gamma_\ell \in G^m_{d,1}$.
  - Construct random walk by $Y_0 = \gamma_1$, $Y_{n+1} = Y_n \gamma_j$, j chosen randomly from $\{1, \ldots, \ell\}$.
  - $\mathrm{supp}(Y_n)$ grows polynomially, not exponentially!
  - The truncated random signature of cubature paths can be considered as such a random walk on $G^m_{d,1}$.
  - The difference between solutions of ODEs driven by recombining cubature paths can be estimated.
  - Degree of the polynomial bound: Hausdorff dimension of group, e.g., degree 4 for $m = 3$, $d = 2$

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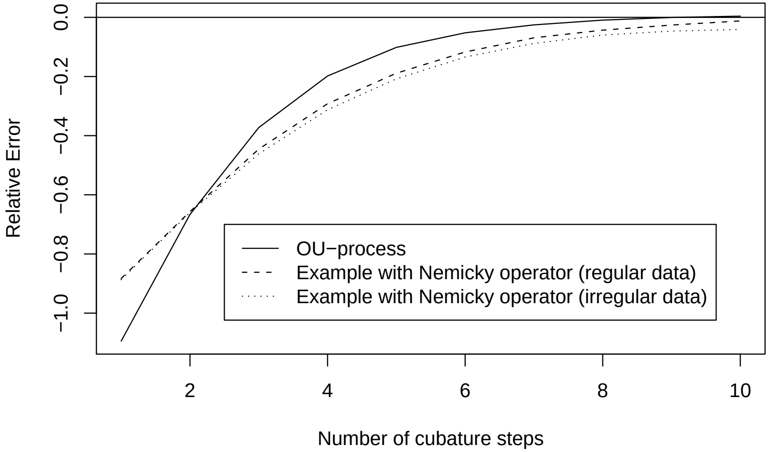
A stochastic heat equation

- $dX_t = \Delta X_t dt + \sin \circ X_t dB_t$
- $H = L^2(]0,1[)$, $\mathcal{D}(\Delta) = H^1_0(]0,1[) \cap H^2(]0,1[)$
- Choose simplest possible cubature formula of degree $m = 3$.
- Note that the equation is non-trivial in the sense that the Stratonovich formulation reads

$$dX_t = \left(\Delta X_t - \frac{1}{2} \cos \circ X_t \sin \circ X_t\right) dt + \sin \circ X_t dB_t$$

Numerical results



An HJM specific implementation

- Given a path $\omega : [0, T] \to \mathbb{R}^d$ of bounded variation and an initial forward rate r_0.
- Consider

$$\begin{cases} dr_t(\omega) = \left(\frac{\partial}{\partial x} r_t + \alpha_{HJM,0}(r_t(\omega))\right)dt + \sum_{i=1}^{d} \sigma_i(r_t(\omega))d\omega_t^i, \\ r_0(\omega) = r_0. \end{cases}$$

- Scheme for solving the above PDE:

$$\bar{r}_{t+\Delta t} = S_{\Delta t}\bar{r}_t + \alpha_{HJM,0}(\bar{r}_t)\Delta t + \sum_{i=1}^{d} \sigma_i(\bar{r}_t)\dot{\omega}_t^i \Delta t,$$

where S_t denotes the shift semigroup.

CIR model

- CIR-model for the short rate Y_t:

$$dY_t = k(\theta - Y_t)dt + \sigma_{CIR}\sqrt{Y_t}dB_t.$$

- Corresponds to HJM model via $r_t(x) = g_0(x) + Y_t g_1(x)$ with

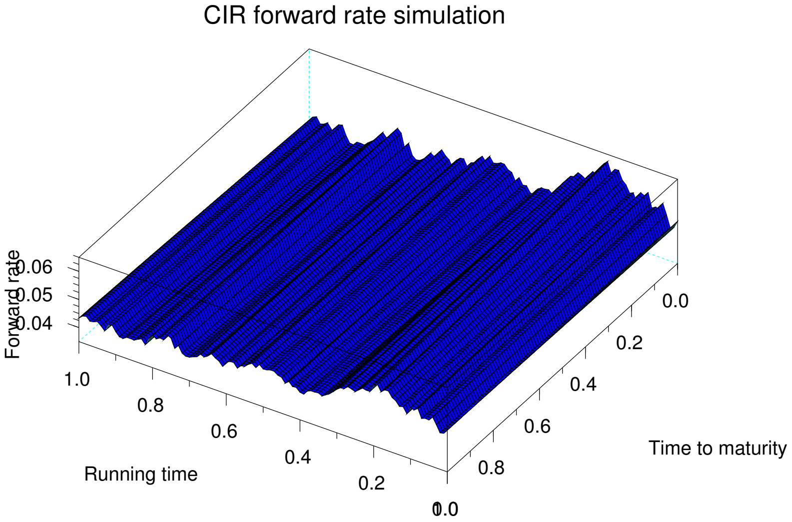
$$g_1(x) = \frac{4\gamma^2 e^{\gamma x}}{\Big((\gamma + k)e^{\gamma x} + \gamma - k\Big)^2}, \quad g_0(x) = k\theta \int_0^x g_1(y)dy,$$

where $\gamma = \sqrt{k^2 + 2\sigma_{CIR}^2}$.

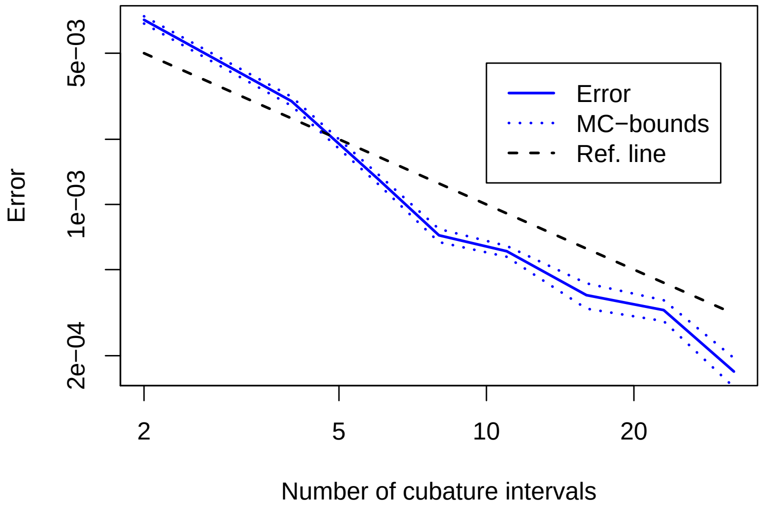
- Satisfies HJM SDE with

$$\sigma(r)(x) = \sigma_{CIR}\sqrt{r(0)}g_1(x).$$

Simulation of HJM-CIR model for $Y_0 = 0.05$



Results for a European Call on a zero coupon bond



Vasiček model

- Vasiček-model for the short rate Y_t:

$$dY_t = k(\theta - Y_t)dt + \sigma_{Vas} dB_t.$$

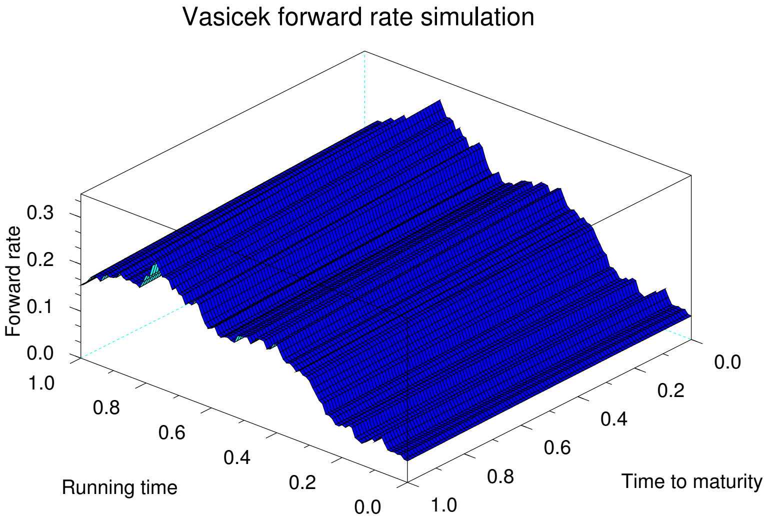
- Corresponds to HJM model via $r_t(x) = g_0(x) + Y_t g_1(x)$ with

$$g_1(x) = e^{-kx}, \quad g_0(x) = k\theta \int_0^x g_1(y)dy - \frac{\sigma_{Vas}^2}{2}\Big(\int_0^x g_1(y)dy\Big)^2.$$

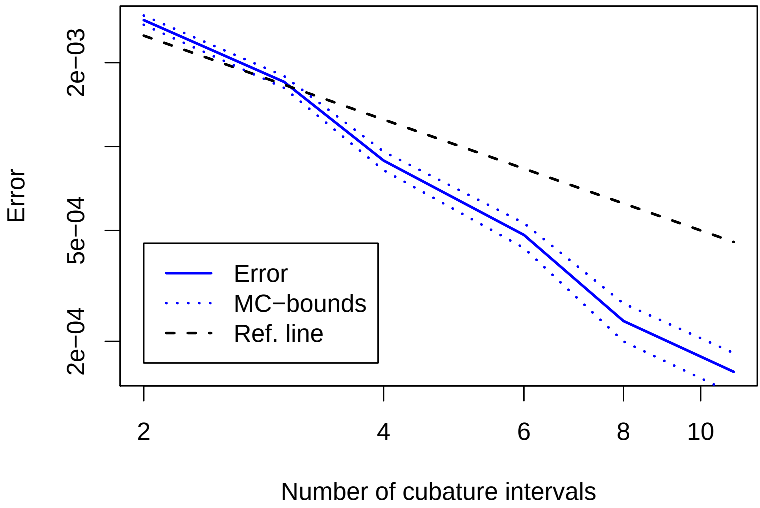
- Satisfies HJM SDE with

$$\sigma(r)(x) = \sigma_{Vas} g_1(x).$$

Simulation of Vasiček model with $Y_0 = 0.05$



Results for a European call on a zero coupon bond



Remarks

- Very good performance for the presented simple benchmark models, but more relevant for much more complicated situations.
- Predicted order of convergence (order 1 for $m = 3$) can be seen in the results.
- Calculations done in Scilab and C (for PREMIA).
- Implementation also includes Bhar-Chiarella-model and a two-factor CIR-model.

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Thank you for your attention!

The space of forward rate curves

Definition

Fix an increasing function $w : [0, \infty[\to [1, \infty[$ such that $w^{-\frac{1}{3}} \in L^1([0, \infty[)$. Define a Hilbert space H_w by

$$H_w = \{h \in L^1_{loc}([0, \infty[) \mid \exists h' \in L^1_{loc}([0, \infty[) \text{ and } \|h\|_w < \infty\}$$

with

$$\|h\|_w = \left|h(0)\right|^2 + \int_0^\infty \left|h'(x)\right|^2 w(x)dx.$$

- H_w consists of continuous functions and the point evaluations $\delta_x(h) = h(x)$ are continuous.
- The right-shift semigroup is a C_0-semigroup on H_w.
- The limit $\lim_{x\to\infty} h(x)$ is well defined for $h \in H_w$. Return