# Obstacle Problems and Optimal Control

## Exercise sheet 5

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Let $K \subset V$ be closed and convex in a Hilbert space $V$ and take $u \in K$.

**1.** Prove that if $\mu \in V^*$ and $u \in K$ satisfy

$$\langle \mu, v - u \rangle \leq 0 \quad \forall v \in K$$

then $\mu \in T_K(u)^\circ$.

**2.** Given $\lambda \in V^*$, prove that

$$\{w \in K - u : \langle \lambda, w \rangle = 0\}^\circ = (R_K(u) \cap \lambda^\perp)^\circ.$$

**3.** Let now $K$ be a closed convex cone and define

$$\mathrm{lin}(u) = \{tu : t \in \mathbb{R}\}$$

to be the linear space generated by $u$.

Show that $R_K(u) = K + \mathrm{lin}(u)$.

**4.** Let $V = H_0^1(\Omega)$ and define

$$W := \{g \in V^* : 0 \leq g \leq F\}$$

where $F \in V^*$ satisfies $F \geq k_0$ for some constant $k_0 > 0$.

Below, by $L^\infty_+(\Omega)$ we mean the set of non-negative elements of $L^\infty(\Omega)$ (and similarly for $L^\infty_-(\Omega)$).

(a) Show that $L^\infty_+(\Omega) \subset R_W(0)$.

(b) Show that $L^\infty_-(\Omega) \subset R_W(F)$.

(c) Now consider a point $f$ such that $0 < c_0 \leq f \leq c_1 < F$ where $c_0$ and $c_1$ are constants. What kind of elements belong to $R_W(f)$?

**5.** Let $F\colon X \to Y$ be a map between Banach spaces.

(a) If $F$ is directionally differentiable and Lipschitz, prove that it is Hadamard differentiable.

(b) If $F$ is Hadamard differentiable, prove that $F'(x)\colon X \to Y$ is continuous.

(c) If $F$ is Hadamard differentiable, prove that the limit

$$\lim_{t \to 0^+} \frac{F(x + th) - F(x)}{t} = F'(x)(h)$$

is uniform in $h$ whenever $h \in C$ belongs to a compact set $C$.

**6.** Define the indicator function of a set $C$ as

$$I_C(x) = \begin{cases} 0 & : x \in C \\ \infty & : x \notin C. \end{cases}$$

Let $A \subset X$ be a non-empty, closed and convex subset of a Banach space and define the function $f\colon X \to \{0, \infty\}$ by $f(x) = I_A(x)$. If $x \in A$, show that $f$ is directionally differentiable at $x$ and characterise its derivative.