Obstacle Problems and Optimal Control

Exercise sheet 3

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1. Prove that the map $M(u) := u^+ = \max(u, 0)$ satisfies $M\colon L^2(\Omega) \to L^2(\Omega)$ and show that it is Lipschitz continuous[1].

2. Let $H := H^1_0(\Omega)$ with the usual inner product

$$(u, v)_H := \int_\Omega \nabla u \cdot \nabla v$$

and define the space

$$V := \left\{ v \in H : \frac{\partial}{\partial x_i}\Delta v \in L^2(\Omega),\ i = 1, \ldots, n \right\}, \qquad \|v\|_V^2 := \|v\|_H^2 + \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i}\Delta v \right\|^2_{L^2(\Omega)}.$$

This forms the Gelfand triple

$$V \subset H \subset V^*$$

(note that the pivot space H is not $L^2(\Omega)$). Consider the bi-Laplace equation for a given $f \in H$:

$$\begin{aligned} \Delta^2 u + u &= f \quad \text{in } \Omega, \\ u = \frac{\partial \Delta u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

(a) By (formally) multiplying by Δv for a test function v, derive a suitable weak formulation and argue well posedness.

(b) Prove that $w := \Delta u$ satisfies

$$\sum_{i=1}^n \int_\Omega \frac{\partial w}{\partial x_i}\frac{\partial z}{\partial x_i} = \int_\Omega (u - f)z \qquad \forall z \in H^1(\Omega).$$

(c) What PDE does this weak form for w correspond to? Make sure to include the boundary condition.

(d) Deduce that u indeed solves (1) in a weak sense.

3. Recall the solution map $S\colon V^* \times V \to V$ defined by

$$S(f, \psi) = u$$

where u solves

$$u \in H^1_0(\Omega),\ u \le \psi : \langle Au - f, u - v\rangle \le 0 \quad \forall v \in H^1_0(\Omega),\ v \le \psi$$

with the usual assumptions on A. Suppose also that A is T-monotone.

(a) Let $f \in V^*$ satisfy $f \ge 0$. Define the sequence

$$\begin{aligned} u_0 &= 0 \\ u_n &= S(f, u_{n-1}). \end{aligned}$$

Prove that $\{u_n\}$ is an increasing sequence, i.e., that $u_n \ge u_{n-1}$ for all n.

[1] An operator $T\colon X \to Y$ between Banach spaces is Lipschitz continuous if there exists a constant $C > 0$ such that

$$\|T(x_1) - T(x_2)\|_Y \le C\,\|x_1 - x_2\|_X.$$

(b) Can you give an interpretation to

$$S(f, \infty)$$

(for $f \in V^*$)? How does this compare to $S(f, \psi)$ (for $\psi \in V$)?

(c) Let $f \in V^*$ and define u_0 via

$$Au_0 = f.$$

Define the sequence

$$u_n = S(f, u_{n-1}).$$

Prove that $\{u_n\}$ is a decreasing sequence.

4. Let $V \subset H \subset V^*$ be a Gelfand triple where $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

(a) Suppose $f, g \in V^*$ and

$$g \leq f \leq g \text{ in } V^*.$$

Using the definition of the dual space inequality, prove that $f = g$.

(b) Suppose we have $u, v, f \in V^*$ such that

$$u \leq f \leq v \quad \text{in } V^*.$$

If $u, v \in H$, prove that $f \in H$ too.

5. Prove the following, which will be used in the next lecture.

(a) If $u \in V := H_0^1(\Omega)$ solves

$$u \leq \psi : \int_\Omega \nabla u \cdot \nabla(u - v) \leq \int_\Omega f(u - v) \quad \forall v \in V,\ v \leq \psi,$$

then for $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$, u also solves

$$u \leq \psi : \int_\Omega \nabla u \cdot \nabla(\varphi(u - v)) \leq \int_\Omega f\varphi(u - v) \quad \forall v \in V,\ v \leq \psi.$$

Hint: consider the two cases $\varphi \not\equiv 0$ and $\varphi \equiv 0$.

(b) Taking further $\varphi \leq 1$, the function $\tilde{u} := \varphi u$ satisfies

$$\tilde{u} \leq \varphi\psi : \int_\Omega \nabla \tilde{u} \cdot \nabla(\tilde{u} - v) \leq \int_\Omega \tilde{f}(\tilde{u} - v) \quad \forall v \in V,\ v \leq \varphi\psi$$

with source term

$$\tilde{f} = \varphi f - u\Delta\varphi - 2\nabla u \nabla\varphi.$$
