Lecture about Efim Zelmanov [1]

***Efim Zelmanov** erhielt die Fields-Medaille für seine brilliante Lösung des lange Zeit offenen Burnside-Problems. Dabei handelt es sich um ein tiefliegendes Problem der Gruppentheorie, der Basis für das mathematische Studium von Symmetrien. Gefragt ist nach einer Schranke für die Anzahl der Symmetrien eines Objektes, wenn jede einzelne Symmetrie beschränkte Ordnung hat. Wie alle bedeutenden mathematischen Resultate haben auch jene von Zelmanov zahlreiche Konsequenzen. Darunter sind Antworten auf Fragen, bei denen ein Zusammenhang mit dem Burnside-Problem bis dahin nicht einmal vermutet worden war. Vor der Lösung des Burnside-Problems hatte Zelmanov bereits wichtige Beiträge zur Theorie der Lie-Algebren und zu jener der Jordan-Algebren geliefert; diese Theorien haben ihre Ursprünge in Geometrie bzw. in Quantenmechanik. Einige seiner dort erzielten Ergebnisse waren für seine gruppentheoritischen Arbeiten von ausschlaggebender Bedeutung. Auf diese Weise wird einmal mehr die Einheit der Mathematik dokumentiert, und es zeigt sich, wie sehr scheinbar weit auseinanderliegende Teilgebiete miteinander verbunden sind und einander beeinflussen.*
Auszug aus der Laudatio von **Walter Feit**.

Efim Zelmanov has received a Fields Medal for the solution of the restricted Burnside problem. This problem in group theory is to a large extent a problem in Lie algebras.

In proving the necessary properties of Lie algebras, Zelmanov built on the work of many others, though he went far beyond what had previously been done. For instance, he greatly simplified Kostrikin's results which settled the case of prime exponent and then extended these methods to handle the prime power case.

The results from Lie algebras which work for exponent p^k with p an odd prime are not adequate for the case of the exponent 2^k. Zelmanov was the first to realize that in this case the theory of Jordan algebras is of great significance.

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Zelmanov had earlier made fundamental contributions to Jordan algebras and was an expert in this area, thus he was uniquely qualified to attack the restricted Burnside problem.

In the sequel, the background from the theory of Jordan algebras and some of Zelmanov's contributions to this theory are first discussed. Then the Burnside problems are described and some of the things that were earlier known about them are listed. Section 4 contains some consequences of the restricted Burnside problem. Finally, some relevant results from Lie and Jordan algebras are mentioned.

1. Jordan Algebras

Jordan algebras were introduced in the 1930's by the physicist P. Jordan in an attempt to find an algebraic setting for quantum mechanics. Hermitian matrices or operators are not closed under the associative product xy, but are closed under the symmetric products $xy + yx$, xyx, x^n. An empirical investigation indicated that the basic operation was the Jordan product

$$x \cdot y = \frac{1}{2}(xy + yx),$$

and that all other properties flowed from the commutative law $x \cdot y = y \cdot x$ and the Jordan identity $(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x)$. Jordan took these as axioms for the variety of Jordan algebras. Algebras resulting from the Jordan product in an associative algebra were called special. In a fundamental paper Jordan, von Neumann, and Wigner classified all finite-dimensional formally-real Jordan algebras. These are direct sums of 5 types of simple algebras: algebras determined by a quadratic form on a vector space and algebras of hermitian $n \times n$ matrices over the reals, complexes, quaternions, and octonions. The algebra of hermitian matrices over the octonions is Jordan only for $n \leq 3$, and is exceptional (= non-special) if $n = 3$, so there was only one exceptional simple algebra in their list (now known as the Albert algebra of dimension 27).

Algebraists developed a rich structure theory of Jordan algebras over fields of characteristic $\neq 2$. First, the analogue of Wedderburn's theory of finite dimensional associative algebras was obtained by Albert. Next this was extended by Jacobson to an analogue of the Wedderburn-Artin theory of semisimple rings with minimum condition. The role of the one-sided ideals was played by inner ideals, defined as subspaces B of the algebra A such that $U_B A \subseteq B$, where $U_x y = 2x \cdot (x \cdot y) - (x \cdot x) \cdot y$. If A is special, then $U_b x = bxb$ in the associative product. Jacobson showed that every non-degenerate Jordan algebra ($U_z = 0$ implies $z = 0$) with d.c.c. on inner ideals is the direct sum of simple algebras which are of classical type.

Up to this point, the structure theory treated only algebras with finiteness conditions because the primary tool was the use of primitive idempotents to introduce coordinates. In 1975 Alfsen, Schultz, and Strömer obtained a Gelfand-Neumark Theorem for Jordan C^*-algebras, and once again the basic structure theorem, but here again it was crucial that the hypoteses guaranteed a rich supply of idempotents.

In three papers (1979–1983), Zelmanov revolutionized the structure theory of Jordan algebras. These deal with prime Jordan algebras, where A is called prime if $U_B C = 0$ for ideals B and C in A, implies that either B or $C = 0$. He proved the remarkable result that a prime non-degenerate Jordan algebra is either special of hermitian type or of clifford type or is a form of the 27-dimensional exceptional algebra. The proofs required the introduction of a host of novel concepts and techniques as well as sharpening of earlier methods e.g. the coordinatization theorem of Jacobson and analogues of the results on radicals due to Amitsur.

One of the consequences of Zelmanov's theory is that the only exceptional simple Jordan algebras, even including infinite dimensional ones, are the forms of the 27-dimensional Albert algebras. Another consequence is that the free Jordan algebra in three or more generators has zero divisors (elements a so that U_a is not injective). This is in sharp contrast to the theorem of Malcev and Neumann that any free associative algebra can be imbedded in a division algebra.

Motivated by applications to analysis and differential geometry, Koecher, Loos, and Meyberg extended the structure theory of Jordan algebras to triple systems and Jordan pairs. Zelmanov applied his methods to obtain new results on these.

To encompass characteristic 2 (which is essential for applications to the restricted burnside problem) it is necessary to deal with quadratic Jordan algebras. These were introduced by McCrimmon and based on the axiomatized properties of the quadratic-linear product $U_a b$.

In a joint paper with McCrimmon, the results on prime algebras were extended to quadratic Jordan algebras.

2. Burnside Problems

A group is *locally finite* if every finite subset generates a finite group. In 1902 W. Burnside studied torsion groups and asked when such groups are locally finite. The most general form is the Generalized Burnside Problem.
(GPB) Is a torsion group necessarily locally finite?
A group G has a *finite exponent* e if $x^e = 1$ for all x in G and e is the smallest natural number with this property. A more restricted version of GPB is the ordinary Burnside Problem.
(BP) Is every group which has a finite exponent locally finite?
There is a universal object $B(r, e)$, (the Burnside group of exponent e on r generators) which is the quotient of the free group on r generators by the subgroup generated by all e^{th} powers. BP is equivalent to
(BP)' Is $B(r, e)$ finite for all natural numbers e and r?
Burnside proved that groups of exponent 2 (trivial) and exponent 3 are locally finite. In 1905 Burnside showed that a subgroup of $GL(n, \mathbf{C})$ of finite exponent is finite. I. Schur in 1911 proved that a torsion subgroup of $GL(n, \mathbf{C})$ is locally finite. This showed that the answers to BP or GBP would necessarily involve groups not discribable in terms of linear transformations over $\mathbf{C}$. Other methods were required.

During the 30's people began to study finite quotients of $B(r, e)$ and considered the following statement.
(RBP) $B(r, e)$ has a unique maximal finite quotient RB(r,e).
W. Magnus called the question of the truth or the falsity of RBP *the restricted Burnside problem.* If such a unique maximal finite quotient $RB(r, e)$ exists for some r and e, then necessarily every finite group on r generators and exponent e is a homomorphic image of $RB(r, e)$. If $RB(r, e)$ exists for some e and all r we say that RBP is true for e.

3. Results

In 1964 E. Golod constructed infinite groups for every prime p, which are generated by 2 elements and in which every element has order a power of p, thus giving a negative answer to GBP. In 1968 S.I. Adian and P.S. Novikov showed that $B(2, e)$ is infinite for e odd and $e > 4380$, thus giving a negative answer to BP.

In a seminal paper P. Hall and G. Higman in 1956 proved a series of results on finite simple groups which, together with the classification of such groups, showed that the truth of RBP will follow once it is proved that $RB(r, p^m)$ exists for all primes p and all natural numbers m and r.

In 1959 I. Kostrikin announced that $RB(r, p)$ exists for a prime p and any natural number r. Kostrikin's argument had some difficulties. He published a corrected and updated version of his proof in his book *Around Burnside (Russian, MR89d, 20032)* which contains numerous references to Zelmanov.

In 1989 Zelmanov announced that RBP is true for all exponents p^m with p any prime, and hence for all exponents by the remarks above. The proof appeared in 1990-91 in Russian. English translations appeared in *Math USSR, Izvestia* 36(1991) 41-60, and *Math USSR Sbornik* 72(1992) 543-564.

4. Some Consequences

This section contains some consequences of RBP. The ideas used in the proof, in addition to the actual result, have also been applied widely.

The next 3 results were proved by Zelmanov as direct consequences of RBP.

Theorem 1. *Every periodic pro-p-group is locally finite.*

Corollary 2. *Every infinite compact (Hausdorff) group contains an infinite abelian subgroup.*

Theorem 3. *Every periodic compact (Hausdorff) group is locally finite.*

Since then, Zelmanov and others, have made several further contributions to the study of pro-p-groups.

5. Lie Algebras

Let G be a finite group of exponent p^k, p a prime. Let $G = G_0$ and $G_{i+1} = [G, G_i]$ for all i. Choose s with $G_s \neq < 1 >$, $G_{s+1} = < 1 >$. Define

$$L(G) = \sum_{i=0}^{s} G_i/G_{i+1}$$

as abelian groups. Then $L(G)$ becomes a Lie ring with $[a_iG_i, a_jG_j] = [a_i, a_j]G_{i+j+1}$, and $L(G)$ has the same nilpotency class as G. Furthermore $L(G)/pL(G)$ is a Lie algebra over $\mathbf{Z}_p$.

A Lie algebra L satisfies the *Engel identity* (E_n) if $ad(x)^n = 0$ for all x in L. An element x in L is *nilpotent* if $ad(x)^n = 0$ for some n. If G has exponent p then L(G) is a Lie algebra over $\mathbf{Z}_p$ which satisfies (E_{p-1}). Kostrikin proved

Theorem 1. *If L is a Lie algebra over $\mathbf{Z}_p$ which satisfies (E_{p-1}) then L is locally nilpotent.*

Theorem 1 implies the Existence of $RB(r,p)$, and so yields RBP for prime exponent. Observe that for prime exponent $e = p$, the case $p = 2$ is trivial, so that it may be assumed $p > 2$. This is in sharp contrast to prime power exponents $e = p^k$, where $p = 2$ is the most complicated case.

Kostrikin called an element a of L a *sandwich* if $[[L,a],a] = 0$ and $[[[L,a],L],a] = 0$. L is a *sandwich algebra* if it is generated by finitely many sandwiches. A first critical result by Kostrikin and Zelmanov is

Theorem 2. *Every sandwich Lie algebra is locally nilpotent.*

Theorem 2 is essential for the proof of Theorem 1.

The main result in Zelmanov's paper on RBP (*Math USSR, Izvestia 1991*) is rather technical but it has the following consequences.

Theorem 3. *Every Lie ring satisfying an Engel condition is locally nilpotent.*

Theorem 4. *$RB(r,p^k)$ exists for p an odd prime.*

Once again an essential part of the proof requires Theorem 2. Let L be a Lie algebra over an infinite field of characteristic p which satisfies an Engel condition. The way to apply Theorem 2 is to construct a polynomial $f(x_1,\dots,x_t)$ that is not identically zero, such that every element in $f(L)$ is a sandwich in L. Actually such a polynomial is not constructed but its existence for $p > 2$ follows only after a very complicated series of arguments. This of course settles RBP for odd exponent.

6. The Case of Exponent 2^k

The outline of the proof of RBP for exponent 2^k is very similar to that for exponent p^k with $p > 2$ described in the previous section. However, the construction of the function f is vastly more complicated. It is here that the quadratic Jordan algebras play an essential role.