Aging for 1D transient RWRE in the sub-ballistic regime

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The model

- **Environment**: $\omega = (\omega_x, \ x \in \mathbb{Z})$ i.i.d. random variables in $(0, 1)$.
  
  $P \equiv \text{law of } \omega$. $E \equiv \text{expectation under } P$. 


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  $P \equiv$ law of $\omega$. $E \equiv$ expectation under $P$.

- $\omega$ fixed, **RWRE**: $X = (X_n, \ n \geq 0)$:
  
  $P_\omega (X_{n+1} = x + 1 \mid X_n = x) = \omega_x, \quad P_\omega (X_{n+1} = x - 1 \mid X_n = x) = 1 - \omega_x.$

$P_\omega \equiv$ law of $X$ in the environment $\omega$ : **quenched law**.
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$$

  $P_\omega \equiv \text{law of } X \text{ in the environment } \omega : \text{quenched law}$.

- $\mathbb{P} \equiv \text{joint law of } (\omega, \ (X_n)) : \text{annealed law}$. $\mathbb{E} \equiv \text{expectation under } \mathbb{P}$.
Transition probabilities

\[
\begin{align*}
1 - \omega_0 & \quad \omega_0 \\
-1 & \quad 0 & \quad 1 \\
\end{align*}
\]

\[
\begin{align*}
1 - \omega_x & \quad \omega_x \\
x - 1 & \quad x & \quad x + 1 \\
\end{align*}
\]
Transience-recurrence criterion

Notations:

\[ \rho_x := \frac{1 - \omega_x}{\omega_x}, \quad x \in \mathbb{Z}. \]
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Theorem (Solomon, 1975)

If \( E[\log \rho_0] \) is defined, \( (X_n, n \geq 0) \) is recurrent iff \( E[\log \rho_0] = 0 \).
Law of large numbers

**Theorem (Solomon, 1975)**

There exists $v \in [-1, 1]$, which depends only on the environment, such that, $\mathbb{P}$-a.s.,

$$\frac{X_n}{n} \longrightarrow v, \quad n \rightarrow \infty,$$
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where $v$ satisfies

$$v := \begin{cases} 
\frac{1-E[\rho_0]}{1+E[\rho_0]} > 0 & \text{if } E[\rho_0] < 1, \\
0 & \text{if } (E[\rho_0^{-1}])^{-1} \leq 1 \leq E[\rho_0], \\
\frac{E[\rho_0^{-1}]-1}{E[\rho_0^{-1}]+1} < 0 & \text{if } 1 < (E[\rho_0^{-1}])^{-1}.
\end{cases}$$
The recurrent case: Sinai’s walk

**Theorem (Sinai, 1982)**

*If* $E[\log \rho_0] = 0$ (*and technical conditions*), then

$$\frac{\sigma^2}{(\log n)^2} \ X_n \xrightarrow{\text{law}} b_\infty ,$$

*where* $\sigma^2 := \text{Var}[\log \rho_0] > 0$. 


Potential

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$$V(x) := \begin{cases} 
\sum_{i=1}^{x} \log \left( \frac{1 - \omega_i}{\omega_i} \right) & \text{if } x \geq 1, \\
0 & \text{if } x = 0, \\
- \sum_{i=x+1}^{0} \log \left( \frac{1 - \omega_i}{\omega_i} \right) & \text{if } x \leq -1.
\end{cases}$$
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\]
Example of potential
Valleys and localization

- **Valleys**: $(a, b, c)$ such that $a < b < c$ and:

  \[
  \min_{a \leq x \leq c} V(x) = V(b),
  \]

  \[
  \max_{a \leq x \leq b} V(x) = V(a),
  \]

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  \max_{b \leq x \leq c} V(x) = V(c).
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- **Height**: \(H = H_{(a,b,c)} := \min(V(c) - V(b), V(a) - V(b))\).
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- **Golosov (1984)**: Exit time $\simeq e^H$. 
Valley and localization in the recurrent case
The sub-ballistic regime

Assumptions

(a) There exists $0 < \kappa < 1$ such that $E[\rho_0^\kappa] = 1$ (and technical conditions).
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Theorem (Kesten-Kozlov-Spitzer, 1975)

Under (a), we have:

\[
\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} c_\kappa S^{ca}_\kappa, \quad n \to \infty,
\]

\[
\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} c'_\kappa \left(\frac{1}{S^{ca}_\kappa}\right)^\kappa, \quad n \to \infty,
\]

where $S^{ca}_\kappa$ is a completely asymmetric stable law of index $\kappa$. 
The sub-ballistic regime

Proof : Branching process in random environment with immigration.

No potential!
Main result: aging phenomenon

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Theorem (Enriquez-Sabot-Z., 2007)

Under assumption (a), we have, for all $h > 1$ and all $\eta > 0$,

$$
\lim_{t \to \infty} \mathbb{P}(|X_{th} - X_t| \leq \eta \log t) = \frac{\sin(\kappa \pi)}{\pi} \int_0^{1/h} y^{\kappa-1}(1 - y)^{-\kappa} \, dy.
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**Remark**

Universality of the Bouchaud’s trap model.
A renewal theorem of Dynkin
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- $(Y_i)_{i \geq 1}$ i.i.d. and heavy tailed: $\mathbb{P}(Y_i \geq u) \sim u^{-\alpha}$, with $\alpha \in (0, 1)$. 
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- **Renewal process** : \(S_n := \sum_{i=1}^{n} Y_i\), for \(n \geq 0\).

- **Last renewal epoch** before time \(t\) defined by

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  N_t := \sup\{n \geq 0 : S_n \leq t\}, \quad t \geq 0.
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- Last renewal epoch before time $t$ defined by

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- Spent waiting time and residual waiting time:

  $$A_t := t - S_{N_t}, \quad t \geq 0,$$

  $$R_t := S_{N_t+1} - t, \quad t \geq 0.$$
A renewal theorem of Dynkin

**Theorem (Dynkin)**

For all $0 \leq x_1 < x_2 \leq 1$, we have

$$
\lim_{t \to \infty} \mathbb{P} \left( x_1 \leq \frac{A_t}{t} \leq x_2 \right) = \frac{\sin(\alpha \pi)}{\pi} \int_{x_1}^{x_2} \frac{x^{-\alpha}}{(1 - x)^{\alpha - 1}} \, dx.
$$

For all $0 \leq x_1 < x_2$, we have

$$
\lim_{t \to \infty} \mathbb{P} \left( x_1 \leq \frac{R_t}{t} \leq x_2 \right) = \frac{\sin(\alpha \pi)}{\pi} \int_{x_1}^{x_2} \frac{\, dx}{x^\alpha (1 + x)}.
$$
The sub-ballistic regime : analysis of the potential
Assumptions

(a) There exists $0 < \kappa < 1$ such that $E \left[ \rho_0^{\kappa} \right] = 1$ (and technical conditions).
Potential

\[ V(x) := \begin{cases} 
\sum_{i=1}^{x} \log \rho_i & \text{if } x \geq 1, \\
0 & \text{if } x = 0, \\
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**Remark** : Assumption (a) implies \( E[\log \rho_0] < 0 \).
Potential and valleys

\[ V(x) \]

\[ n_t := t^k \]
Potential and valleys

- Excursions of the potential above its past minimum

\[
e_0 := 0, \\
e_i := \inf\{n > e_{i-1} : V(n) \leq V(e_{i-1})\}, \quad i \geq 1.
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• \((V(x) - V(e_{i-1}), e_{i-1} \leq x \leq e_i)_{i \geq 1}\) are i.i.d.
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- Iglehart’s result: \(P\{H > h\} \sim C_1e^{-\kappa h}, \quad h \to \infty\).
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- Deep valleys: boxes constructed around excursions higher than \(h_t := \log t - \log \log t\).
Potential and valleys

\[ V(x) \]

\[ N = N(t) \]
Valleys’ properties

- “Directed” property.
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- The time spent between deep valleys is **negligible**:

\[ \tau(d_N) \simeq \tau(b_1, d_1) + \tau(b_2, d_2) + \cdots + \tau(b_N, d_N). \]
Valleys’ properties

- “Directed” property.
- The time spent between deep valleys is negligible:
  \[ \tau(d_N) \approx \tau(b_1, d_1) + \tau(b_2, d_2) + \cdots + \tau(b_N, d_N). \]
- The valleys are well separated: “i.i.d.” property.
Occupation time

- **Height**: $H_k := V(c_k) - V(b_k)$, for $k \geq 1$. 
Occupation time

- **Height**: \( H_k := V(c_k) - V(b_k) \), for \( k \geq 1 \).

- **Exact computation**: \( \forall \lambda > 0 \),

\[
E_\omega \left[ e^{-\lambda \tau(b_k,d_k)} \right] \approx \frac{1}{1 + \lambda e^{H_k} \overline{M}_k \overline{M}_k},
\]

where

\[
\overline{M}_k := \sum_{i=a_k}^{c_k} e^{-(V(i)-V(b_k))},
\]

\[
\overline{M}_k := \sum_{i=b_k}^{d_k} e^{V(i)-V(c_k)}.
\]
A renewal theorem

The sub-ballistic regime

Model and result

**Occupation time**

\[ V(x) \]

\[ H_k \]

\[ M_k \] := \sum_{i=b_k}^{d_k} \text{e}^{V(i) - V(c_k)}

\[ \overline{M}_k \] := \sum_{i=a_k}^{c_k} \text{e}^{-(V(i) - V(b_k))}

**Fig.:** \( M_k \) et \( \overline{M}_k \).
Properties

- Occupation time: asymptotically (quenched result)

\[
\tau(b_k, d_k) \xrightarrow{\text{law}} (M_k \overline{M}_k e^{H_k}) \exp\{1\}.
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- **Occupation time**: asymptotically (quenched result)

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\tau(b_k, d_k) \overset{\text{law}}{\approx} (\underline{M}_k \overline{M}_k e^{H_k}) \exp\{1\}.
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- **Asymptotic independence** between \(e^{H_k}, \underline{M}_k\) and \(\overline{M}_k\): coupling arguments.
Properties

- Occupation time: asymptotically (quenched result)

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- Asymptotic independence between \( e^{H_k}, \underline{M}_k \) and \( \overline{M}_k \): coupling arguments.

- Iglehart’s result + \( \underline{M}_k \) and \( \overline{M}_k \) “nice” r.v. \( \Rightarrow \tau(b_k, d_k) \) is heavy tailed under the annealed law.
Proof

- $\tau(b_1, d_1) + \tau(b_2, d_2) + \cdots + \tau(b_N, d_N)$ sum of “i.i.d.” heavy-tailed random variables.
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- Occupation time: $T_i := \tau(b_i, d_i)$. 
Proof

- \( \tau(b_1, d_1) + \tau(b_2, d_2) + \cdots + \tau(b_N, d_N) \) sum of “i.i.d.” heavy-tailed random variables.

- Occupation time : \( T_i := \tau(b_i, d_i) \).

- Time between deep valleys negligible + “directed” property :

  \[
  \{ a_j \leq X_t \leq d_j \} = \left\{ \sum_{i=1}^{j-1} T_i \leq t < \sum_{i=1}^{j} T_i \right\}
  \]
Proof

• Last visited deep valley: $\ell_t := \sup\{ j \geq 0 : \tau(b_j) \leq t \}$. 
Proof

- Last visited deep valley: \( \ell_t := \sup\{j \geq 0 : \tau(b_j) \leq t\} \).
- As for renewal processes:

\[
\{a_{\ell_t} \leq X_t, X_{th} \leq d_{\ell_t}\} = \left\{ \sum_{i=1}^{\ell_t-1} T_i \leq t < th < \sum_{i=1}^{\ell_t} T_i \right\}
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• New version of Dynkin’s theorem!
Proof

- Residual waiting time:

\[
\left\{ \sum_{i=1}^{\ell_t-1} T_i \leq t < th < \sum_{i=1}^{\ell_t} T_i \right\} = \left\{ \frac{R_t}{t} \geq h - 1 \right\}
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\[ \left\{ \sum_{i=1}^{\ell_t-1} T_i \leq t < th < \sum_{i=1}^{\ell_t} T_i \right\} = \left\{ \frac{R_t}{t} \geq h - 1 \right\} \]

• Then, we have, when \( t \to \infty \),

\[ \mathbb{P}(a_{\ell_t} \leq X_t, X_{th} \leq d_{\ell_t}) \to \frac{\sin(\kappa \pi)}{\pi} \int_0^{1/h} y^{\kappa-1}(1 - y)^{-\kappa} \, dy. \]
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• Control around the bottom of the last visited deep valley: arguments of invariant measure for a Markov chain on a finite state space + geometrical properties of the valleys.