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Institutskolloquium

Principle of Linearized Stability and Invariant Manifold Theorem for Semilinear Hyperbolic Systems with Applications to Semiconductor Laser Dynamics
Overview

▶ Introduction
  ▶ Principle of linearized stability
  ▶ Growth and spectral bound, exponential dichotomy for linearized dynamical systems
  ▶ Center manifold theorem

▶ Linearization of semilinear hyperbolic systems
  ▶ The variation of constants formula
  ▶ Sun star calculus

▶ Stability and dichotomy for linearized hyperbolic systems
  ▶ Proof of principle of linearized stability and center manifold theorem for semilinear hyperbolic systems

▶ Applications
Given a dynamical system

\[ \frac{d}{dt} x = f(x). \]

- **State** \( x \in X \)
- **ODE:**
  - \( X = \mathbb{R}^n \)
  - \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) ist \( C^k \) smooth
- **PDE:**
  - \( X \) Banach-space
  - \( f \) a densely defined operator
Determining the stability of stationary states

Let $x_0$ be a stationary state.

1. Linearize in $x_0$:
   \[
   \frac{d}{dt} h = Df(x_0) h.
   \]

2. Determine the stability of the linearized problem:
   ▶ Locate the spectrum of $Df(x_0)$.

3. Prove that the nonlinear problem is stable near $x_0$.

**Theorem (Principle of linearized stability)**

Suppose there exists $s < 0$, so that for all $\lambda \in \sigma(Df(x_0))$

\[
\Re \lambda \leq s < 0.
\]

Then $x_0$ is exponentially stable.
Approximation of the nonlinear dynamics via the linearized dynamics

- For the proof we need that the linearization $Df(x_0)$ is a good approximation for $f$ near $x_0$.

- PDE: The operator $f$ contains nonlinear Nemytskij operators. Their differentiability properties depend on the topology of the Banach-space $X$.
  - Usually it is not enough to consider only one Banach space $X$. Often we need a triple or even scale of Banach spaces.
Stability of the linearized problem

- As is well known in finite dimensions the stability of the linear system \[
\frac{d}{dt} h = Df(x_0)h
\] is determined by the eigenvalues (spectrum) of the matrix \( Df(x_0) \).

- In infinite dimensions, where \( X \) is a Banach-space, the issue is more complex.

- The appropriate abstract setting is provided by the theory of \( C_0 \) semigroups \( (e^{At})_{t \geq 0} \) of bounded linear operators on the Banach-space \( X \).
Growth and spectral bound

Definition

Let $A = Df(x_0)$ be a generator of a $C_0$ semigroup $e^{At}$. The spectral bound $s(A)$ is defined as

$$s(A) := \sup \{ \Re z \mid z \in \sigma(A) \}.$$ 

The growth bound $\omega(A)$ is per definitionem

$$\omega(A) := \inf \{ \omega \in \mathbb{R} \mid \exists M = M(\omega) > 0 : \|e^{At}\| \leq Me^{\omega t} \text{ for } t \geq 0 \}.$$ 

- $\omega(A) = s(A)$ for ODEs, DDEs, semilinear parabolic PDEs.
- In general: $\omega(A) \geq s(A)$, equality must not hold.
- Warning: There exists a counterexample of a 2d wave equation with $\omega(A) > s(A)$. 
Determining the growth bound $\omega(A)$

**Proposition**

For $t > 0$

$$\omega(A) = \frac{\log r(e^{At})}{t},$$

where $r(e^{At}) := \sup \{|z| \mid z \in \sigma(e^{At})\}$ denotes the spectral radius of the semigroup $e^{At}$.

Method for determining the growth bound $\omega$:

- Calculate $\sigma(A)$ by solving spectral problem.
- Important open question for hyperbolic PDEs: Can the unknown spectrum $\sigma(e^{At})$ of the semigroup be calculated from the spectrum $\sigma(A)$ of the generator $A$ (the equations of the PDE)?

**Theorem**

For hyperbolic systems in 1d the answer is positive: $\omega(A) = s(A)$. 
Existence of center manifolds

Assumptions:

- $s(A) \leq 0$
- $E_c := \sigma(A) \cap i\mathbb{R} \neq \emptyset$
- Spectral gap: There exists $\delta > 0$ such that

$$\{ z \in \mathbb{C} \mid -\delta < \Re z < 0 \} \subset \rho(A).$$

Let $\pi_c : X^C \to X^C$ denote spectral projection corresponding to the critical eigenvalues $E_c$, where $X^C$ denotes complexification of $X$. Further let

$$X_c := X \cap \text{Im}(\pi_c) = X \cap \bigoplus_{\lambda \in E_c} \bigcup_{j=1}^{\infty} \text{Ker} \left( \lambda \text{Id} - A \right)^j,$$

$$X_s := X \cap \text{Ker}(\pi_c).$$
Spectrum of the traveling wave operator (LDSL)

Eigenvalues of $H(N)$

$2\text{Im}(\Omega)$ vs. Relative wavelength, nm

Linearized Stability and Invariant Manifold Theorem for Semilinear Hyperbolic Systems
For ordinary differential equations, delay equations and semilinear parabolic PDEs it is known, that the spectral gap condition generates an exponential dichotomy on the spectral decompositions.

- Let $T_c(t) := e^{At}_{|X_c}$ and $T_s(t) := e^{At}_{|X_s}$
- $\exists c > 0 : \|T_s(t)\| \leq ce^{-\delta t}$ for $t \geq 0$
- $\forall \epsilon > 0 \exists d > 0 : \|T_c(-t)\| \leq de^{\epsilon t}$

Exponential dichotomy is necessary for the proof of center manifold theorem.

- If there is no exponential dichotomy, it is known due to a result of Mane, that the critical eigenspace $X_c$ does not persist under small nonlinear perturbations.
Center manifold theorem

**Theorem**

There exists a neighbourhood $U$ of zero in $X$ and a smooth graph $\gamma : X_c \cap U \rightarrow X_s$ with the following properties:

- the manifold $M := \{x_0 + x_c + \gamma(x_c) \mid x_c \in U \cap X_c\}$ is locally invariant and exponentially attractive with respect to the nonlinear semiflow,
- any solution $u : \mathbb{R} \rightarrow x_0 + U$ lies on $M$,
- the trajectories on $M$ are governed by the equation

$$\frac{d}{dt}x_c = Df(x_0)x_c + \pi_c r(x_c + \gamma(x_c)),$$

where the remainder $r$ is of order 2.
Core problems for existence of invariant manifolds in infinite dimensional dynamical systems

- Does a spectral gap generate an exponential dichotomy?
- Does the evolution equation form a smooth semiflow on $X$? Is the solution map linearizable with respect to the norm of $X$?
- If yes, for which Banach-spaces are both properties fulfilled?

These issues have been resolved for large classes of semilinear parabolic PDEs and DDEs, but not for hyperbolic PDEs.
A general class of semilinear hyperbolic systems

\[
\begin{align*}
\text{(SH)} \quad \left\{ \begin{array}{l}
\frac{\partial}{\partial t} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} + H(x, u(t, x), v(t, x)), \\
v(t, l) = D u(t, l), \\
u(t, 0) = E v(t, 0).
\end{array} \right.
\end{align*}
\]

\begin{itemize}
\item $x \in ]0, l[,$ $t > 0$
\item $u \in \mathbb{R}^{n_1}, v \in \mathbb{R}^{n_2}, n = n_1 + n_2,$ $D, E$ matrices
\item $K(x) = \text{diag} \ (k_j(x))_{1 \leq j \leq n}, k_j \in C^1([0, l], \mathbb{R}),$
\end{itemize}

\[
 k_j < 0 \quad 1 \leq j \leq n_1, \quad k_j > 0 \quad n_1 + 1 \leq j \leq n.
\]

\begin{itemize}
\item $H : ]0, l[ \times \mathbb{R}^n \to \mathbb{R}^n$ smooth in $(u, v)$, measurable in $x$
\end{itemize}
Variation of constants formula

Let $T(t)$ be the reflection / shift semigroup generated by

$$
\begin{cases}
    \frac{\partial}{\partial t} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix}, \\
    u(t, 0) = E v(t, 0), \quad v(t, l) = D u(t, l) \\
    u(0, x) = u_0(x), \quad v(0, x) = v_0(x).
\end{cases}
$$

The nonlinearity $H : ]0, l[ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ generates a Nemytskij operator: For $u : ]0, l[ \rightarrow \mathbb{R}^{n_1}$, $v : ]0, l[ \rightarrow \mathbb{R}^{n_2}$

$$
\mathcal{H} (u, v) (x) := H(x, u(x), v(x)).
$$

Formally the variation of constants formula for (SH) reads

$$
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T(t - s) \mathcal{H}(u(s), v(s)) \, ds.
$$
Which choice of space $X$?

\[
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T(t - s) \mathcal{H}(u(s), v(s)) \, ds.
\]

It is tempting to take the Hilbert space $L^2([0, l], \mathbb{R}^n)$ for $X$:

- $T(t)$ is strongly continuous on $L^2$.
- The Nemytskij operator $\mathcal{H}$ is not well defined on $L^2$.
  - Need to truncate the nonlinearity $H$ so that the Nemytskij operator becomes well defined and globally Lipschitz on $L^2$.
- But still it is not Fréchet differentiable due to the (rather surprising) fact that $\mathcal{H} : L^2 \to L^2$ is differentiable at some $(u, v) \in L^2([0, l], \mathbb{R}^n)$ if and only if for almost all $x \in [0, l]$ the function $z \mapsto H(x, z)$ is affine.
A good choice for $X$

\[ \left( \begin{array}{c} u(t) \\ v(t) \end{array} \right) = T(t) \left( \begin{array}{c} u_0 \\ v_0 \end{array} \right) + \int_0^t T(t - s) H(u(s), v(s)) \, ds. \]

Take

\[ X := \left\{ (u, v) \in C([0, l], \mathbb{R}^n) \mid u(0) = Ev(0), \quad v(l) = Du(l) \right\}. \]

- $T(t)$ is strongly continuous on $X$.
- But $H$ maps $X$ out to a larger space: If $(u, v) \in X$ then $H(u, v) \notin X$ for almost any choice of $H$ and $(u, v)$.
- Need to enlarge the space $X$!
Enlarging $X$, the sun star space

Idea: Construct a larger space in terms of a combination of properties of the space $X$ and the semigroup $T$.

- Let $T^*(t) : X^* \to X^*$ be the adjoint semigroup
- Then $t \to T^*(t)x^*$ is not necessarily continuous (even not Bochner measurable, but weak star continuous). Let

$$X^{\Diamond} := \left\{ x^* \in X^* \mid \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0 \right\}$$

be the subspace on which $T^*$ is strongly continuous.

- Define $j : X \to X^{\Diamond,*}$, $\langle jx, x^{\Diamond} \rangle := \langle x^{\Diamond}, x \rangle$ ($X^{\Diamond,*} := (X^{\Diamond})^*$)
- $j$ is injective since $X^{\Diamond}$ is weak star dense in $X^*$, hence

$$X \overset{j}{\hookrightarrow} X^{\Diamond,*}.$$
Enlarging $X$, the sun star space

Put

$$T^\circ(t) := T^*(t)|_{X^\circ}.$$ 

By definition $T^\circ(t) : X^\circ \to X^\circ$ is a strongly continuous semigroup.

Again consider the adjoint semigroup $T^{\circ*}(t) = (T^\circ(t))^* : X^{\circ*} \to X^{\circ*}$.

$$\forall x^\circ \in X^\circ : \langle T^{\circ*}(t)jx, x^\circ \rangle = \langle jx, T^\circ(t)x^\circ \rangle$$

$$= \langle T^\circ(t)x^\circ, x \rangle$$

$$= \langle x^\circ, T(t)x \rangle$$

$$= \langle j(T(t)x), x^\circ \rangle.$$ 

▶ Hence $j(T(t)x) = T^{\circ*}(t)jx$ or

$$j \circ T(t) = T^{\circ*}(t) \circ j.$$
Enlarging $X$, the sun star space

\[
T(t) : X \quad \longrightarrow \quad T^*(t) : X^*
\]

\[
T^{\bigcirc*}(t) : X^{\bigcirc*} \quad \longleftarrow \quad T^{\bigcirc}(t) : X^{\bigcirc}
\]

\[
X \quad \xrightarrow{T(t)} \quad X
\]

\[
\downarrow j \quad \quad \downarrow j
\]

\[
X^{\bigcirc*} \quad \xrightarrow{T^{\bigcirc*}(t)} \quad X^{\bigcirc*}
\]
The sun star space for hyperbolic systems with reflection boundary conditions

**Theorem**

\( X^{\odot*} \) is isomorphic to \( L^\infty([0, l], \mathbb{R}^n) \).

For \( (u, v) \in X \):

\[
T^{\odot*}(t) \begin{pmatrix} u \\ v \end{pmatrix} = T(t) \begin{pmatrix} u \\ v \end{pmatrix}.
\]

The main advantages of using the space \( X \) together with its sun dual \( X^{\odot*} \) are based on the following two Lemmas:

**Lemma**

If \( H(x, z) \) is measurable with respect to \( x \) and smooth with respect to \( z \) then the Nemytskij operator \( \mathcal{H}(u, v)(x) := H(x, u(x), v(x)) \) is a smooth map from \( X \) into \( X^{\odot*} \).
Variation of constants formula

Moreover we get back from $X^\circledast$ into the small space $X$:

**Lemma**

Let $f : [0, T] \to X^\circledast$ be norm continuous. Then the weak-star integral

$$ t \mapsto \int_0^t T^\circledast(t - s)f(s) \, ds $$

is norm continuous and takes values in $X$.

**Definition (Variation of constants formula)**

$(u, v) \in C([0, T], X)$ is called a mild (or weak) solution to $(SH)$ if

$$
\begin{pmatrix}
  u(t) \\
  v(t)
\end{pmatrix} = T(t) \begin{pmatrix}
  u_0 \\
  v_0
\end{pmatrix} + \int_0^t T^\circledast(t - s) \mathfrak{H}(u(s), v(s)) \, ds.
$$
Some straightforward consequences

**Theorem (Unique local existence)**

For any \((u_0, v_0) \in X\) there exists a \(\delta > 0\), depending only on 
\[\|(u_0, v_0)\|_X, \mathcal{H}\text{ and } T(t),\] such that \((SH)\) has a unique mild solution 
\[(u, v) \in C([0, \delta], X)\] with \(u(0) = u_0, v(0) = v_0\).

**Theorem**

Let \(z \in C([0, T], X)\) be a weak solution of \((SH)\). Then there exists a
neighborhood \(U\) of \(z(0)\) in \(X\) such that for all \(y_0 \in U\) there is a weak
solution \(y \in C([0, T], X)\) of \((SH)\) satisfying \(y(0) = y_0\).

There exists a constant \(c > 0\) such that for all \(y_0 \in U\)

\[
\|z(t) - y(t)\|_X \leq c\|z(0) - y_0\|_X.
\]
The smooth semiflow

Suppose there exists a weak solution \( z \in C([0, T], X) \) of (SH). Then according to the last Theorem there exists an open neighborhood \( U \) of \( z(0) \) in \( X \) so that we can define a solution map

\[
S^t : U \rightarrow X, \quad S^t(y_0) := y(t) \quad (t \in [0, T]).
\]

Theorem (Smooth semiflow property)

For each \( t \in [0, T] \) the map \( S^t : U \rightarrow X \) is \( C^k \) smooth. The map \( (t, u) \mapsto S^t u \) is continuous from \( [0, T] \times U \) into \( X \). The total derivative

\[
DS^t, \begin{pmatrix} \tilde{h}_u(t) \\ \tilde{h}_v(t) \end{pmatrix} = DS^t \begin{pmatrix} h_u \\ h_v \end{pmatrix}
\]

satisfies the equation

\[
\begin{pmatrix} \tilde{h}_u(t) \\ \tilde{h}_v(t) \end{pmatrix} = T(t) \begin{pmatrix} h_u \\ h_v \end{pmatrix} + \int_0^t T^{\circ \ast}(t - s) D\mathcal{H}(u(s), v(s)) \begin{pmatrix} \tilde{h}_u(s) \\ \tilde{h}_v(s) \end{pmatrix} ds.
\]
Linearization of \((\text{SH})\)

Let \((u_0, v_0)\) be a stationary state. Then the last theorem states that the linearized flow \(DS^t(u_0, v_0)\) is given by the mild solutions to the linearized system

\[
\begin{cases}
    \frac{\partial}{\partial t} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} \\
    + \partial_{(u,v)} H(x, u_0(x), v_0(x)) \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix},
\end{cases}
\]

\[
v(t,l) = D u(t,l), \quad u(t,0) = E v(t,0).
\]

**Proposition**

The linearized flow \(DS^t(u_0, v_0)\) is a \(C_0\) semigroup \(e^{At}\) on \(X\) with infinitesimal generator

\[
A \begin{pmatrix} u \\ v \end{pmatrix} = K(x) \frac{\partial}{\partial x} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} + \partial_{(u,v)} H(x, u_0(x), v_0(x)) \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}.
\]
Let $\alpha < \beta$. A has a $(\alpha, \beta)$ exponential dichotomy, if there exists a projection $\pi : X^{\mathbb{C}} \rightarrow X^{\mathbb{C}}$ such that

- $\pi e^{At} = e^{At} \pi$
- For $T_1(t) := e^{At}_{|\text{Im} (\pi)}$ and $T_2(t) := e^{At}_{|\text{Ker}(\pi)}$
  - $\omega(T_1(t)) \leq \alpha$
  - $T_2(t)$ extends to a group with $\omega(T_2(-t)) \leq -\beta$. 
\((\alpha, \beta)\) exponential dichotomy

\[ \forall \delta > 0 \ \exists \ c > 0 \ \forall \ t \geq 0 \]

\[ \| T_1(t) \| \leq c e^{(\alpha + \delta)t} \]

\[ \| T_2(t) \| \geq \frac{1}{c} e^{(\beta - \delta)t} \]
Characterization of exponential dichotomy

**Theorem**

The following are equivalent:

- $A$ has a $(\alpha, \beta)$ exponential dichotomy.
- $\forall t > 0 : \{ \lambda \in \mathbb{C} \mid e^{\alpha t} < |\lambda| < e^{\beta t} \} \subset \rho(e^{At})$.
- $\exists t_0 > 0 : \{ \lambda \in \mathbb{C} \mid e^{\alpha t_0} < |\lambda| < e^{\beta t_0} \} \subset \rho(e^{At_0})$.

- Exponential dichotomy means that there is a circular spectral gap for the semigroup $e^{At}$.
- Does a spectral gap condition on $A$ imply the presence of a circular spectral gap for $e^{At}$?
Spectral mapping theorems for linearized hyperbolic systems

**Theorem**

- $\sigma(e^{At}) \setminus \{0\} = e^{\sigma(A)t} \setminus \{0\}$ in $L^2([0, l], \mathbb{C}^n)$
- $\ln X^C = \{(u, v) \in C([0, l], \mathbb{C}^n) \mid u(0) = Ev(0), v(l) = Du(l)\}$ for all $\alpha < \beta$ and $t > 0$ we have

  $$\{z \in \mathbb{C} \mid \alpha < \Re z < \beta\} \subset \rho(A)$$

  $\iff \{z \in \mathbb{C} \mid e^{\alpha t} < |z| < e^{\beta t}\} \subset \rho(e^{At})$.

**Corollary**

*If* $\alpha < \beta$ and $\{\lambda \in \mathbb{C} \mid \alpha < \Re \lambda < \beta\} \subset \rho(A)$, *then* $A$ *has a* $(\alpha, \beta)$ *exponential dichotomy.*
Spectral mapping theorem \( \sigma(e^{A t}) = e^{\sigma(A) t} \)
Proof of spectral mapping theorem

- High frequency estimates of spectrum and resolvent parallel to the imaginary axis:
  - For high frequencies spectrum and resolvent are approximated by the diagonal system.
  - For \( \lambda \) on stripes in the resolvent set parallel to the imaginary axis we have for \( |\text{Im} \lambda| \) sufficiently large

\[
R(\lambda, A) = R(\lambda, A_0) + \frac{1}{\lambda} R_1(\lambda) + O\left(\frac{1}{\lambda^2}\right).
\]

- \( A_0 \) denotes the diagonal system, obtained by cancelling all nondiagonal entries in the linearized differential equation. Since equations decouple there is a closed formula for \( R(\lambda, A_0) \).
- Error term \( R_1(\lambda) \) as well as higher order terms can be calculated recursively (terms quite complicated).
Estimates for spectrum

**Theorem**

There exists an exponential polynomial $h_0$ and an entire (characteristic) function $h$ with the following properties:

- $\sigma(A) = \{ \lambda \in \mathbb{C} \mid h(\lambda) = 0 \}$,
- $\sigma(A_0) = \{ \lambda \in \mathbb{C} \mid h_0(\lambda) = 0 \}$,
- For all $r > 0$ there exist $c, d > 0$ such that for all $\lambda \in \mathbb{C}$ with $|\Re \lambda| < r$ und $|\Im \lambda| > d$ we have:
  \[
  \left| h(\lambda) - h_0(\lambda) - \frac{1}{\lambda} h_1(\lambda) \right| \leq c \frac{1}{|\lambda|^2},
  \]
- There is a closed formula for $h_1$ (quite complicated).
Resolvent estimates

**Theorem**

Let $U \subset \rho(A)$ be such that

$$
\sup_{\lambda \in U} |\Re \lambda| < \infty, \quad \inf_{\lambda \in U} |h_0(\lambda)| > 0.
$$

Then there exists $d > 0$ such that for $\lambda \in U$ with $|\Im \lambda| \geq d$

- $R(\lambda, A) = R(\lambda, A_0) + \frac{1}{\lambda} R_1(\lambda, A) + \frac{1}{\lambda^2} E(\lambda, A),$
- $R(\lambda, A_0), R_1(\lambda, A)$ and $E(\lambda, A)$ are bounded on $U,$
- There are closed formulas for $R_1(\lambda, A)$ and $R(\lambda, A_0).$

- In particular the resolvent $R(\lambda, A)$ is bounded on $U.$
By applying an important theorem due to Gearhart/Herbst/Prüss [Trans. AMS 1984] the resolvent estimates imply the following spectral mapping property for linearized hyperbolic systems in the Hilbert space $L^2$

$$
\sigma(e^{At}) \setminus \{0\} = e^{\sigma(A)t} \setminus \{0\}.
$$

Problem: theorem of Gearhart/Herbst/Prüss requires Hilbert-space.

The semiflow is not strongly linearizable in $L^2$.

We need a spectral mapping theorem or characterization of exponential dichotomy in terms of the spectrum of $A$ in the smaller nonreflexive Banach-space $X^C$. 

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Spectral mapping theorem in $L^2$

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Spectral mapping property in non Hilbert-space $X^C$

- For Banach-space the situation is more difficult. Counterexamples show that Gearhart/Herbst/Prüss spectral mapping theorem fails, in general.

- Idea: Use $C_1$ Laplace-inversion formula. For $\rho > \omega(A)$

$$e^{At}x = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} e^{zt} R(z, A)x \, dz$$

$$:= \frac{1}{2\pi} \lim_{R \to \infty} e^{\rho t} \int_{-R}^{R} e^{i\nu t} R(\rho + i\nu, A)x \left(1 - \frac{|\nu|}{R}\right) \, d\nu.$$

- Works also in non Hilbert-space $X^C$!
Characterization of \((\alpha, \beta)\) dichotomy in \(X^C\)

**Theorem (1994 Lunel, Kaashoek in J. Diff. Eq.)**

A has a \((\alpha, \beta)\) exponential dichotomy if and only if

1. \(\rho(A) \supset \{\lambda \in \mathbb{C} \mid \alpha < \Re \lambda < \beta\}\),
2. For all \(\delta > 0\):
   \[\sup_{\alpha + \delta < \Re \lambda < \beta - \delta} \| R(\lambda, A) \| < \infty,\]
3. For all \(\rho \in ]\alpha, \beta[\) there exists a constant \(K_\rho > 0\) such that for all \(x \in X^C, x^* \in (X^C)^*\)
   \[\mathcal{F}r(\cdot, \rho, x, x^*) \in L^\infty(\mathbb{R}) \text{ and } \| \mathcal{F}r(\cdot, \rho, x, x^*) \|_{L^\infty} \leq K_\rho \| x \| \| x^* \|,\]
   where \(r(\cdot, \rho, x, x^*) : \mathbb{R} \rightarrow \mathbb{C}\) is defined as
   \[r(\nu, \rho, x, x^*) := x^* R(\rho + i\nu, A)x.\]
Characterization of $\left(\alpha, \beta\right)$ dichotomy in $X^C$

- Necessity of conditions follows directly from the Hille-Yosida theorem and the $C_1$ Laplace inversion formula applied to $A|\text{Im }\pi$ and $A|\text{Ker }\pi$.
- The resolvent estimates are in sufficiently closed form so that the Fourier transforms can be estimated.
- Warning: Convergence of improper Fourier integrals only in Cesaro mean $C_1$, no absolute convergence.
- Tools:
  - $\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} \mathcal{F}^{-1} f(\omega) \, d\omega = \frac{f(t+) + f(t-)}{2}$,
  - Wiener Algebra property of absolutely convergent Fourier series.

Theorem

*Principle of linearized stability and center manifold theorem hold true for hyperbolic systems.*
The results are applicable to large classes of practical problems:

- Stability and bifurcation analysis in Laser dynamics
- Model Reduction: Mode approximations [Bandelow, Wenzel, Wünsche 1993]
- Turing-Models with correlated random walk [Kac, Goldstein, Hadeler, Hillen, Horsthemke, ...]
- Boltzmann-systems
- Tubular reactor processes
- Systems of vibrating strings
- Differential equations with delay
- ...
The traveling wave model

\[
\frac{1}{v_g} \partial_t E^\pm = \left( \mp \partial_z - i \beta(n) \right) E^\pm - i \kappa E^\mp - \frac{\bar{g}}{2} (E^\pm - P^\pm)
\]

\[
\partial_t P^\pm = \bar{\gamma} (E^\pm - P^\pm) + i \omega P^\pm
\]

\[
\partial_t n = I - R(n) - v_g \Re \langle E, g(n) E - \bar{g}(E - P) \rangle_{\mathbb{C}^2}
\]

\[
E^+(t, 0) = r_0 E^-(t, 0) + \alpha(t), \quad E^-(t, l) = r_l E^+(t, l).
\]

d t \in \mathbb{R} \text{ time, } z \in [0, l] \text{ longitudinal coordinate}

d E^\pm = E^\pm(t, z) \in \mathbb{C} \text{ complex envelope of optical field, } P^\pm = P^\pm(t, z) \in \mathbb{C} \text{ polarization, } n = n(t, z) \in \mathbb{R} \text{ carrier density}

d spontaneous recombination \( R(n) = An + Bn^2 + Cn^3 \)

d propagation coefficient \( \beta(n) = \delta_0 - i \frac{\alpha}{2} + \frac{i}{2} g(n) + \delta_N(n) \)

d field gain \( g(n) = G' \log \frac{n}{n_{tr}} \), effective index dependence \( \delta_N(n) = -\sqrt{n'n} \), current injection \( I = I(t, z) \), optical injection \( \alpha(t) \) at left facet of laser, reflection coefficients \( r_0 \) and \( r_l \)

d all coefficients depend on lateral coordinate \( z \)
A two parameter numerical bifurcation analysis of the raveling wave model (LDSL, M. Radziuniunas)
Current Work: The 2d diffraction extended traveling wave model

\[
\frac{1}{v_g} \partial_t E^\pm = -i \frac{1}{2K} \partial_{xx} E^\pm + (\mp \partial_z - i\beta(n)) E^\pm - i\kappa E^\mp - \frac{g^2}{2} (E^\pm - P^\pm)
\]

\[
\partial_t P^\pm = \gamma (E^\pm - P^\pm) + i\omega P^\pm
\]

\[
\partial_t n = d_n \partial_{xx} n + I - R(n) - v_g \Re \langle E, g(n)E - \bar{g}(E - P) \rangle C^2
\]

\[
E^+(t, 0, x) = r_0(x) E^-(t, 0, x) + \alpha(t, x), \quad E^-(t, l, x) = r_l(x) E^+(t, l, x).
\]

▷ All coefficients now depend on longitudinal \((z)\) and lateral \((x)\) coordinate.
▷ In optical equation for \(E\) a diffraction operator has been added (red).
▷ In carrier equation for \(n\) a lateral diffusion operator has been added (blue).
Stable pulsating high power diode

Laser geometry and parameters provided by FBH.